Contributions to the Geometry of Lorentzian Manifolds with Special Holonomy

DISSERTATION

zur Erlangung des akademischen Grades

doctor rerum naturalium
(Dr. rer. nat.)

eingereicht an der
Mathematisch-Naturwissenschaftlichen Fakultät der
Humboldt-Universität zu Berlin

von
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Eingereicht am: 14.10.2014

Verteidigt am: 20.03.2015

ABSTRACT

In the present thesis we study $(n+2)$-dimensional Lorentzian manifolds $(\mathcal{M}^{(n+2)}, g)$ with *special holonomy*, i.e. such that their holonomy representation acts indecomposably but non-irreducibly. Being indecomposable, their holonomy group leaves invariant a degenerate subspace $W \subset T_x\mathcal{M}$ and thus a light-like line $L = W \cap W^{\perp_g}$. Geometrically, this means that, since being holonomy invariant, this line gives rise to ∇^g-parallel subbundles $\mathbb{L}$ and $\mathbb{L}^\perp$ of the tangent bundle $T\mathcal{M}$, where ∇^g denotes the Levi-Civita connection to g. In particular this implies that these subbundles induce foliations of dimension resp. codimension one. Hence we naturally gain a link to foliation theory and by the following observations also to Riemannian geometry. Given $\mathbb{L}$ and $\mathbb{L}^\perp$ we can define by $\Sigma := \mathbb{L}^\perp/\mathbb{L}$ a vector bundle over $\mathcal{M}$ which is called the *screen bundle* and equip it with a connection ∇^Σ induced by ∇^g. It is well-known that the holonomy of this bundle w.r.t. ∇^Σ coincides with the $\mathrm{O}(n)$-projection of the full holonomy group of $(\mathcal{M}^{(n+2)}, g)$ and that this in turn is a holonomy group of a Riemannian manifold. Moreover, given a (non-canonical) *screen distribution* $\mathbb{S} \subset T\mathcal{M}$ isomorphic to Σ, one can associate to $\mathbb{S}$ a Riemannian metric g^R which coincides with g on $\mathbb{S} \times \mathbb{S}$. This thesis makes use of these naturally given objects on a Lorentzian manifold with special holonomy to prove the following insights.

In the first chapter we address the problem of finding conditions under which a compact Lorentzian manifold is geodesically complete, a property, which always holds for compact Riemannian manifolds. It is known that a compact Lorentzian manifold is geodesically complete if it is homogeneous, or has constant curvature, or admits a time-like conformal vector field. We consider certain Lorentzian manifolds with Abelian holonomy, which are locally modeled by the so called pp-waves, and which, in general, do not satisfy any of the above conditions. We show that compact pp-waves are universally covered by a vector space, determine the metric on the universal cover, and prove that they are geodesically complete. Using this, we show that every Ricci-flat compact pp-wave is a plane wave.

The second chapter is devoted to the study of the topology and geometry of certain Lorentzian manifolds with special holonomy and high first Betti number. Namely, assuming compactness of the leaves to $\mathbb{L}^\perp$ and non-negative Ricci curvature on these leaves it is known that the first Betti number is bounded by the dimension of the manifold resp. the leaves, if the manifold is compact or non-compact. We prove in the case of the maximality of the first Betti number that every such Lorentzian manifold is – up to finite cover – diffeomorphic to the torus (in the compact case) or the product of the real line with a torus (in the non-compact case) and has very degenerate curvature, i.e. the curvature tensor induced on the leaves is light-like.

The last chapter turns the attention to the investigation of geometric properties of indecomposable but non-irreducible Lorentzian manifolds, which are total spaces of circle bundles. We investigate under which conditions these manifolds are complete and give examples which fulfill the obtained conditions. In particular we investigate the Einstein equation for these spaces yielding examples for complete compact Ricci flat Lorentzian manifolds and manifolds with timelike Killing vector fields. Finally we study their holonomy and obtain in particular complete examples for Lorentzian manifolds with holonomy of so called type 4.

ZUSAMMENFASSUNG

In dieser Arbeit studieren wir $(n+2)$-dimensionale Lorentz-Mannigfaltigkeiten $(\mathcal{M}^{(n+2)}, g)$ mit *spezieller Holonomie*, d.h. ihre Holonomiedarstellung wirkt schwach-irreduzibel aber nicht irreduzibel. Aufgrund der schwachen Irreduzibilität, lässt die Darstellung einen ausgearteten Unterraum $W \subset T_x\mathcal{M}$ invariant, damit also auch eine lichtartige Linie $L = W \cap W^{\perp_g}$. Geometrisch hat dies zur Folge, dass wir zwei ∇^g-parallele Unterbündel $\mathbb{L}$ und $\mathbb{L}^\perp$ des Tangentialbündels erhalten, wobei ∇^g den Levi-Civita Zusammenhang zu g bezeichnet. Insbesondere induzieren $\mathbb{L}$ und $\mathbb{L}^\perp$ aufgrund ihrer Parallelität Blätterungen der Dimension eins bzw. Kodimension eins auf $\mathcal{M}$. Dies schlägt eine Brücke zur Blätterungstheorie und vermöge der folgenden Beobachtungen ebenfalls zur Riemannschen Geometrie. Definieren wir durch $\Sigma := \mathbb{L}^\perp/\mathbb{L}$ ein Vektorbündel über $\mathcal{M}$ und nennen dieses *Screenbündel*, so können wir dazu einen Zusammenhang ∇^Σ definieren, der durch ∇^g induziert wird. Es ist wohlbekannt, dass die Holonomie von Σ bzgl. ∇^Σ mit der $\mathrm{O}(n)$-Projektion der Holonomiegruppe von $(\mathcal{M}^{(n+2)}, g)$ übereinstimmt, wobei diese wiederum die Holonomie einer Riemannschen Mannigfaltigkeit ist. Des Weiteren können wir uns eine (nicht kanonische) *Screendistribution* $\mathbb{S} \subset T\mathcal{M}$ isomorph zu Σ vorgeben und zu dieser eine Riemannsche Metrik g^R assoziieren, welche mit g auf $\mathbb{S} \times \mathbb{S}$ übereinstimmt. Die vorliegende Arbeit nutzt diese Beobachtungen, um die nachfolgend genannten Erkenntnisse über Lorentz-Mannigfaltigkeiten zu beweisen.

Im ersten Kapitel beschäftigen wir uns mit der geodätischen Vollständigkeit von kompakten Lorentz-Mannigfaltigkeiten. Diese Eigenschaft gilt für kompakte Riemannsche Mannigfaltigkeiten immer und für kompakte Lorentz-Mannigfaltigkeiten sind bekannte hinreichende Bedingungen für geodätische Vollständigkeit die Homogenität, konstante Krümmung oder die Existenz eines zeitartigen konformen Vektorfeldes. Wir studieren in dieser Arbeit Lorentz-Mannigfaltigkeiten mit abelscher Holonomie. Diese sind lokal gegeben als sogenannte pp-Wellen und im Allgemeinen erfüllen sie keine der soeben genannten Bedingungen. Wir zeigen, dass kompakte pp-Wellen geodätisch vollständig sind, universell von $\mathbb{R}^{n+2}$ überlagert werden und beschreiben die auf die Überlagerung zurückgezogene Metrik. Unter Zuhilfenahme dieser Resultate zeigen wie schließlich, dass jede kompakte Ricci-flache kompakte pp-Welle eine ebene Welle ist.

Das zweite Kaptitel widmet sich dem Studium der Topologie und Geometrie von bestimmten Lorentz-Mannigfaltigkeiten mit spezieller Holonomie und hoher erster Bettizahl. Unter der Annahme der Kompaktheit der Blätter zu $\mathbb{L}^\perp$ und nicht-negativer Ricci-Krümmung auf den Blättern ist bekannt, dass diese durch die Dimension der Mannigfaltigkeit bzw. die Dimension der Blätter nach oben beschränkt ist, abhängig davon, ob die Mannigfaltigkeit kompakt ist oder nicht. Im Gleichheitsfall zeigen wir, dass jede solche Lorentz-Mannigfaltigkeit – bis auf endliche Überlagerung – diffeomorph zum Torus (im kompakten Fall) bzw. zum Produkt von $\mathbb{R}$ mit dem Torus (im nicht-kompakten Fall) ist. Zudem hat g in diesem Fall einen sehr ausgearteten Krümmungstensor, genauer gesagt ist die Krümmung auf den Blättern zu $\mathbb{L}^\perp$ lichtartig.

Im letzen Kapitel widmen wir unsere Aufmerksamkeit dem Studium geometrischer Eigenschaften von Lorentz-Mannigfaltigkeit mit spezieller Holonomie, welche als Totalräume über $\mathbb{S}^1$-Bündeln definiert sind. Wir untersuchen unter welchen Voraussetzungen diese geodätisch vollständig sind und geben Bedingungen an, unter denen die Konstruktion Ricci-flache Lorentz-Mannigfaltigkeiten zulässt. Zuletzt geben wir geodätisch vollständige Beispiele für Lorentz-Mannigfaltigkeiten mit sogenannter Holonomie vom Typ 4.

Dedicated to Sina and Pia.

CONTENTS

INTRODUCTION

An important concept in differential geometry is that of holonomy. Given a semi-Riemannian manifold[1] $(\mathcal{M}^{(n+2)}, g)$ (i.e. whose metric g has arbitrary signature (p,q)) we define the (reduced) holonomy group $\mathrm{Hol}^{(0)}_x(\mathcal{M}^{(n+2)}, g)$ in a point $x \in \mathcal{M}$ as the group of parallel displacements along (null-homotopic) loops closed in $x \in \mathcal{M}$. As a consequence the holonomy group is contained in the orthogonal group $\mathrm{O}(T_x\mathcal{M}, g_x) \simeq \mathrm{O}(p,q)$ and thus gives a natural representation ρ of $\mathrm{Hol}_x(\mathcal{M}, g)$ on $\mathrm{O}(T_x\mathcal{M}, g_x)$. Holonomy groups are smooth Lie groups and as such they have a corresponding Lie algebra $\mathfrak{hol}_x(\mathcal{M}^{(n+2)}, g)$, the *holonomy algebra* which, by the Holonomy Theorem of Ambrose and Singer, provides a description of the curvature of the manifold by algebraic means. In this spirit, the concept of holonomy is of high importance in the study of semi-Riemannian manifolds since it links algebraic and geometric properties allowing to apply results from algebra as to obtain geometric results. Moreover, holonomy is in close relation to parallel sections in geometric vector bundles on the manifold by the holonomy principle which relates elements stabilized by the holonomy group to parallel sections. Further, for manifolds with *special holonomy* – by which we mean that the holonomy group is a proper subgroup of $\mathrm{O}(p,q)$ – one can deduce special geometric properties such as for the curvature. In the Riemannian case, the study of special holonomy groups provided the starting point for applications of holonomy theory, where E. Cartan in the 20's and finally Berger in the 50's obtained remarkable and groundbreaking results. Indeed, Berger obtained a classification of the Riemannian holonomy groups in [Ber55]. The list contains the possible irreducible holonomy groups of simply-connected, not locally-symmetric n-dimensional Riemannian manifolds and is commonly referred to as *Berger's list*. It contains exactly the groups $\mathrm{SO}(n)$, $\mathrm{U}(\frac{n}{2})$, $\mathrm{SU}(\frac{n}{2})$, $\mathrm{Sp}(\frac{n}{4})$, $\mathrm{Sp}(\frac{n}{4}) \cdot \mathrm{Sp}(1)$, G_2 and $\mathrm{Spin}(7)$. Moreover, if the holonomy of a Riemannian manifold is contained in one of these groups, this has consequences on their geometry by which they are Kähler-, Calabi-Yau-, hyper-Kähler-, quaternionic Kähler-, G_2- or $\mathrm{Spin}(7)$-manifolds, respectively.

For indefinite metrics this classification problem is widely open, except for the Lorentzian case, where quite recent results lead to a classification of Lorentzian manifolds with special holonomy. However, for the case that the holonomy representation ρ is *irreducible* meaning that there exists no proper holonomy invariant subspace $E \subset T_x\mathcal{M}$, such a classification exists [Ber55, Ber57] even for the semi-Riemannian case. The difficulties in indefinite signature arise since in this case the holonomy representation ρ can also be both, non-irreducible and *weakly-irreducible*, where ρ is called *weakly-irreducible* (or *indecomposable*) if there there is no proper *non-degenerate* holonomy invariant subspace. By the de Rham/Wu decomposition theorem [DR52, Wu64], any simply-connected, geodesically complete semi-Riemannian manifold is globally isometric to a product

1 Within this thesis, all manifolds are assumed to be smooth, connected and without boundary.

of a flat manifold (which is possibly zero dimensional) and indecomposable non-flat manifolds. Hence, for indefinite metrics, the classification of manifolds with special holonomy breaks down to classify the indecomposable ones.

For every Lorentzian manifold whose reduced holonomy group is a proper subgroup of $\mathrm{SO}^0(1, n+1)$ the holonomy representation acts indecomposably but non-irreducibly [DSO01]. Consequently, its holonomy representation ρ needs to preserve a degenerate subspace $W \subset T_x\mathcal{M}$ and hence a holonomy invariant light-like line $L := W \cap W^{\perp_g}$. By the holonomy principle, the holonomy group must lie in the stabilizer $\mathrm{SO}^0(1, n+1)_L$ of this line. Based on results in [BBI93] in which the possible subalgebras of the Lie algebra $\mathfrak{so}(1, n+1)_L$ of this stabilizer were determined algebraically, the connected components of indecomposable Lorentzian holonomy groups were finally classified by Leistner in [Lei07]. In addition to that Galaev [Gal06] gave a construction method for Lorentzian metrics and proved that all possible groups occurring in the classification can actually be realized as holonomy groups of a Lorentzian manifold. We point out that holonomy groups of four-dimensional Lorentz spaces where classified much earlier [Sch61, Sha70]. A very nice survey about the general classification is given in [LG08] and an overview concerning more recent results about holonomy groups of Lorentzian manifolds in [Bau12].

Motivation and Open Problems

Due to Ikemakhen and Bérard-Bergery [BBI93], the indecomposable, non-irreducible subalgebras of $\mathfrak{so}(1, n+1)_L$ can algebraically be only of four types.

Theorem I. *Let $\mathfrak{h} \subset \mathfrak{so}(1, n+1)_L$ be a weakly-irreducible subalgebra and let $\mathfrak{g} := \mathrm{pr}_{\mathfrak{so}(n)}(\mathfrak{h})$ denote the orthogonal part. Then $\mathfrak{h}$ belongs to one of the following types:*

Type 1: $\mathfrak{h} = (\mathbb{R} \oplus \mathfrak{g}) \ltimes \mathbb{R}^n$,

Type 2: $\mathfrak{h} = \mathfrak{g} \ltimes \mathbb{R}^n$,

Type 3: $\mathfrak{h} = \{(\varphi(X), X + Y, z) \mid X \in \mathfrak{z}(\mathfrak{g}),\ Y \in [\mathfrak{g}, \mathfrak{g}], z \in \mathbb{R}^n\}$, where $\varphi : \mathfrak{z}(\mathfrak{g}) \twoheadrightarrow \mathbb{R}$ is a surjective homomorphism,

Type 4: $\mathfrak{h} = \{(0, X + Y, \varphi(X) + z) \mid X \in \mathfrak{z}(\mathfrak{g}),\ Y \in [\mathfrak{g}, \mathfrak{g}], z \in \mathbb{R}^k\}$, where $\mathbb{R}^n = \mathbb{R}^m \oplus \mathbb{R}^k$, $0 < m < n$, $\mathfrak{g} \subset \mathfrak{so}(k)$ and $\varphi : \mathfrak{z}(\mathfrak{g}) \twoheadrightarrow \mathbb{R}^m$ is a surjective homomorphism.

By the classification theorem of Leistner and Galaev [Lei07, Gal06], these are actually all realizable as holonomy algebras of Lorentzian manifolds, whereas the orthogonal part $\mathfrak{g} = \mathrm{pr}_{\mathfrak{so}(n)}(\mathfrak{hol}(\mathcal{M}, g))$ is the holonomy algebra of a *Riemannian* manifold.

Theorem II. *Let $H \subset \mathrm{SO}^0(1, n+1)$ be a connected subgroup acting indecomposably but non-irreducibly. Then H is the reduced holonomy group of a Lorentzian manifold if and only if its orthogonal part is a Riemannian holonomy group.*

Although this gives a complete understanding on how possible holonomy groups of indecomposable but non-irreducible Lorentzian manifolds can algebraically be described, it is a – in contrast to the Riemannian case – widely open field to understand implications on the *geometry* of a Lorentzian manifolds with special holonomy if one assumes that its holonomy belongs to a certain prescribed type.

In light of the results in the Riemannian case and the fact that the orthogonal part of the holonomy group of a Lorentzian manifold with special holonomy coincides with a holonomy group of a Riemannian manifold it seems to be worthwhile to understand this correspondence in more detail. Namely, note that for Lorentzian manifolds $(\mathcal{M}^{(n+2)}, g)$ with special holonomy, the light-like line L in each $(T_x\mathcal{M}, g_x)$ gives rise to a parallel line bundle $\mathbb{L}$ via parallel translation and hence a parallel codimension one subbundle $\mathbb{L}^\perp$ since $\mathbb{L} \subset \mathbb{L}^\perp$. Consider the quotient bundle $\Sigma := \mathbb{L}^\perp/\mathbb{L}$, which is usually referred to as the *screen bundle* together with its bundle metric $\langle \cdot, \cdot \rangle_\Sigma$ and connection ∇^Σ, naturally induced by the metric g and its Levi-Civita connection. Then the orthogonal part of the holonomy group of $(\mathcal{M}, g)$ coincides with the holonomy[2] of (Σ, ∇^Σ) [Lei06, BLL14]. Consequently, geometric data encoded in the screen bundle translates into algebraic data of the holonomy and vice versa. For example, this correspondence can be used to give conditions for the existence of parallel spinors on $(\mathcal{M}, g)$ [Kat99, Bau02, Lei07, BLL14]. Another observation which relates the geometry of a Lorentzian manifold with special holonomy to algebraic properties can be found in [LG08] where it was proven that the holonomy of a Lorentzian Einstein manifold with special holonomy has to have holonomy of type 1 or 2 in the classification theorem and, moreover, the orthogonal part $\mathfrak{g}$ must be either trivial or a combination of $\mathfrak{so}(k)$, $\mathfrak{su}(k)$, $\mathfrak{sp}(k)$, $\mathfrak{g}_2$, $\mathfrak{spin}(7)$ and the holonomy algebra of a non-Kählerian Riemannian symmetric space. In light of these results we may pose the following problem.

Problem A. *Find examples for Lorentzian manifolds with prescribed geometry (e.g. Einstein, Ricci-flat, parallel spinors etc.)* and *prescribed holonomy.*

A screen distribution $\mathbb{S} \subset T\mathcal{M}$ is an n-dimensional subbundle of the tangent bundle of a Lorentzian manifold $(\mathcal{M}^{(n+2)}, g)$ with special holonomy, isomorphic to the screen bundle Σ. The choice of such a screen distribution is not unique and indeed the understanding of Lorentzian manifolds relies heavily on the comprehension of the existence of certain screen distributions. For example, by fixing a screen distribution $\mathbb{S}$ on a time-orientable[3] Lorentzian manifold we can define an associated Riemannian metric g^R to this screen by letting $Z \in \Gamma(T\mathcal{M})$ be a light-like vector field with $g(V, Z) = 1$ such that $\mathbb{S} = V^{\perp_g} \cap Z^{\perp_g}$ and define

$$g^R(V, \cdot) := g(Z, \cdot),\ g^R(Z, \cdot) := g(V, \cdot), g^R(X, \cdot) := g(X, \cdot) \text{ for } X \in \Gamma(\mathbb{S})$$

with extension by linearity. This Riemannian metric on $\mathcal{M}$ enables us to relate methods from Riemannian geometry to the Lorentzian manifolds with special holonomy and as we will see later, its properties are somehow related to properties of the underlying screen distribution.

2 The holonomy of a geometric vector bundle $(\Sigma, \nabla^\Sigma, \langle \cdot, \cdot \rangle)$ over $\mathcal{M}$ is defined similar to the holonomy group of a semi-Riemannian manifold $(\mathcal{M}, g)$ but by replacing the parallel displacement w.r.t. the Levi-Civita connection of g by the connection ∇^Σ.

3 A Lorentzian manifold $(\mathcal{M}, g)$ is said to be time-orientable if and only if it admits a nowhere vanishing timelike vector field X, i.e. with $g(X, X) < 0$.

The importance of g^R becomes more clear if we recall that, being parallel, the distributions $\mathbb{L}$ and $\mathbb{L}^\perp$ induce a one- and codimension one foliation on $\mathcal{M}$, respectively. If $\mathcal{L}$ resp. $\mathcal{L}^\perp$ denote leaves of these foliations, and if we denote with $\mathscr{F}$ the foliation induced by a vector field V spanning $\mathbb{L}$ then the metric g^R equips any leaf $\mathcal{L}^\perp$ corresponding to $\mathbb{L}^\perp$, foliated by $\mathscr{F}$, with a special geometric structure turning the triple $(\mathcal{L}^\perp, \mathscr{F}, g^R)$ into a *Riemannian flow*. Hence we obtain an intersection between Riemannian geometry, foliation theory and Lorentzian geometry. That this interplay is fruitful has already been shown in [Lär11] (see also the following paragraphs) and it raises the following question.

Problem B. *Understand the interplay between screen distributions, induced foliations and the geometry of Lorentzian manifolds with special holonomy.*

As we have already mentioned, the curvature of the manifold has a close relation to the holonomy algebra by the theorem of Ambrose and Singer. This relation manifests in the following result [Lei06, Lei07] concerning Lorentzian manifolds with Abelian holonomy: assume that the parallel line bundle $\mathbb{L}$ is spanned by a global section V or, equivalently, that the Lorentzian manifold is time-orientable. Then the Lorentzian manifold has holonomy $\mathbb{R} \ltimes \mathbb{R}^n$ if and only if the curvature satisfies $R^g|_{\mathbb{L}^\perp \wedge \mathbb{L}^\perp} = 0$. If the section V is in addition parallel, then this curvature condition is equivalent to an Abelian holonomy, i.e. to be equal to $\mathbb{R}^n$. In particular, $R^g|_{\mathbb{L}^\perp \wedge \mathbb{L}^\perp} = 0$ is equivalent for the screen bundle to have trivial holonomy. Throughout this thesis we refer to Lorentzian manifolds with parallel light-like vector field and Abelian holonomy as *pp-waves*. Four-dimensional pp-waves were discovered by Brinkmann in the context of conformal geometry [Bri25], and then played an important role in general relativity (e.g., see [EK62], where also the name *pp-wave* for *plane fronted with parallel rays* was introduced). More recently, higher dimensional pp-waves appeared in supergravity theories, e.g. in [Hul84], and there is now a vast physics literature on them.

However, beside curvature there are also other geometric questions which are interesting to investigate, for example the property of geodesic completeness. Recall that, in sharp contrast to the Riemannian signature, *compact* Lorentzian manifolds do *not* have to be complete. The probably most popular counterexample is the Clifton-Pohl torus, which is compact, but geodesically incomplete [O'N83, Example 7.16]. However, quite a long list of results investigating completeness of Lorentzian manifolds exists, e.g. for the case of the existence of timelike conformal Killing fields [Kam93, RS94a, RS94b, RS95] and for *general plane fronted waves* [CFS03, CRS13, CRS12], see also [Sán13] for an overview. Imposing strong assumptions, a *compact* Lorentzian manifold is complete, for example, if it is flat [Car89][4], has constant curvature [Kli96], or if it is homogeneous. In fact, Marsden proved in [Mar73] that any compact homogeneous semi-Riemannian manifold is complete. Moreover, compact, *locally* homogeneous 3-dimensional Lorentzian manifolds are complete [DZ10]. Finally, we should mention the Lorentzian symmetric spaces, the *Cahen-Wallach spaces* [CW70] which are, as symmetric spaces, automatically complete. We subsume these observations to the following problem.

4 In fact, in [Car89] Carrière proved a much more general result for affine manifolds. A direct proof for the flat case was given in [Yur92]. However, this proof has gaps as it was pointed out in [RS93].

Problem C. *For the possible holonomy types 1 - 4 find geometric implications on the metric (e.g. geodesic (in-)completeness, curvature conditions etc.). In particular, find conditions for the holonomy of a compact Lorentzian manifold to be geodesically complete.*

It is a well-known result due to WALKER [Wal50] that, locally, every Lorentzian manifold $(\mathcal{M}^{(n+2)}, g)$ with special holonomy is isometric to an open neighborhood $\mathcal{U}$ in which the metric g takes the form

$$g|_{\mathcal{U}} = 2dudv + 2Hdu^2 + 2\sum_{i=1}^{n} A_i dx_i du + \sum_{i,j=1}^{n} h_{ij} dx_i dx_j, \tag{$\mathscr{W}$}$$

with $A_i, h_{ij} \in C^\infty(\mathcal{U})$ s.t. $\frac{\partial A_i}{\partial v} = \frac{\partial h_{ij}}{\partial v} = 0$ and $H \in C^\infty(\mathcal{U})$. In these coordinates, the parallel line bundle $\mathbb{L}$ is, locally, spanned by ∂_v. An interesting question concerns the globalization of this result in the following sense.

Problem D. *Under which assumptions can it be shown that a Lorentzian manifold $(\mathcal{M}^{(n+2)}, g)$ with special holonomy is (universally) covered by $\Phi : \mathbb{R}^2 \times \mathcal{N} \longrightarrow \mathcal{M}$ with the metric $\widetilde{g} := \Phi^* g$ being isometric to a metric of the form ($\mathscr{W}$)?*

Lorentzian manifolds with holonomy of type 1 or 2 and a prescribed orthogonal part are easier to find than examples for the types 3 and 4 where there is a coupling between the $\mathbb{R}$- and the $\mathbb{R}^n$-part with the $\mathfrak{so}(n)$-part, respectively. Indeed, on the one hand (even compact or complete) examples with trivial topology providing all possible connected holonomy groups of type 1 or 2 can be constructed quite easily [Lei02]. On the other hand, examples for Lorentzian manifolds with holonomy of type 3 or 4 are very rare and the only examples we know are [Gal06, Baz09, Lei06]. In fact, neither are these compact, nor is it known if they provide geodesically complete examples. In particular, all examples are of the form $\mathcal{L}_1 \times \mathcal{L}_2 \times \mathcal{N}$ for some manifold $\mathcal{N}$ and $\mathcal{L}_i \in \{\mathbb{R}, \mathbb{S}^1\}$.

Problem E. *Find (geodesically complete) examples (with non-trivial topology) for Lorentzian manifolds with special holonomy, especially for the holonomy types 3 and 4.*

By imposing further geometric assumptions on an arbitrary Lorentzian manifold with special holonomy one can investigate possible implications on the topology of the underlying manifold. For example, this was done in [Lär11] where the assumption of non-negative Ricci curvature on the leaves of $\mathbb{L}^\perp$ caused bounds for the Betti numbers of the underlying manifold. This brings us to the following problem.

Problem F. *Find connections between Lorentzian manifolds with special holonomy and the topology of the underlying manifold.*

Finally, let us mention a special class of pp-waves, called *plane waves*. On their curvature it is imposed as another condition that $\nabla^g R^g = V^\flat \otimes Q$ for some $(4,0)$-tensor Q. These, at a first glance, play an important role in physics literature (see e.g. [EK62, BO03, Bla09] among others) but also appear in more general mathematical contexts as Lorentzian manifolds with *essentially parallel Weyl tensor* (or *essentially conformally symmetric manifolds*). These have parallel Weyl tensor but are neither conformally flat nor locally symmetric and were studied by DERDZIŃSKI and ROTER. In fact, recent results [DR09, DR08, DR10] study global properties and especially prove that they are plane wave metrics. In the physics literature [EK62], however, there was posed a pure mathematical problem to which we will refer to as *Ehlers-Kundt problem*.

Problem G (Ehlers-Kundt [EK62, Section 2-5.7])**.** *Prove the plane waves to be the only g-complete pp-waves, no matter which topology one chooses.*

We stress that in the Ehlers-Kundt problem, pp-waves are understood to be solutions of the Einstein vacuum field equations and hence, in addition are assumed to be Ricci flat.

Outline of the Thesis

This thesis is divided into four chapters, the last three of which presenting basically the articles [Sch12, LS13, Sch13, Sch14]. The main results obtained therein are contained in the Theorems VIII, IX, XIII, XIV, XVI and Theorem XVII stated below.

In Chapter 1 we will present the necessary preliminary results used throughout this thesis. However, we do not claim completeness of this exposition and will refer for more elaborated introductions into the presented topics to the sections itself. In Section 1.1 we will provide a short overview about well-known results concerning Riemannian and Lorentzian holonomy theory, albeit the last stated theorem in this section is a quite recent result which provides a tool to compute the full holonomy group of Lorentzian manifolds $(\mathcal{M}^{(n+2)}, g)$ with special holonomy.

Theorem III ([BLL14, Theorem 3])**.** *Let $(\mathcal{M}^{(n+2)}, g)$ be a Lorentzian manifold with special holonomy universally covered by $(\widetilde{\mathcal{M}}, \widetilde{g})$ with $\widetilde{\mathcal{M}} = \mathbb{R}^2 \times \mathcal{N}$ and $\widetilde{g}$ as in* (1.13)*. Then each isometry σ of $(\widetilde{\mathcal{M}}, \widetilde{g})$ is of the form*

$$\sigma(u, v, p) = (a_\sigma^{-1} u + b_\sigma, a_\sigma v + \tau_\sigma(u, v, p), \nu_\sigma(u, v, p))$$

with[5] *$a_\sigma \in \mathbb{R}^*, b_\sigma \in \mathbb{R}, \tau_\sigma \in C^\infty(\widetilde{\mathcal{M}})$ with $\partial_v(\tau_\sigma) = 0$ and smooth $\nu_\sigma : \widetilde{\mathcal{M}} \longrightarrow \mathcal{N}$ such that $\partial_v(\nu_\sigma) = 0$ and $\nu(u, v, \cdot)$ is an isometry of $(\mathcal{N}, h)$ for all $u, v \in \mathbb{R}$. Then we find*

$$\mathrm{Hol}_x(\mathcal{M}^{(n+2)}, g) = Q \cdot \mathrm{Hol}_{\widetilde{x}}(\widetilde{\mathcal{M}}^{(n+2)}, \widetilde{g}) = Q \cdot \mathrm{Hol}_x^0(\mathcal{M}^{(n+2)}, g),$$

where $\Phi : \widetilde{\mathcal{M}} \longrightarrow \mathcal{M}$ denotes the universal covering, $\widetilde{x} = (u, v, p)$, $\Phi(\widetilde{x}) = x$, and

$$Q := \langle Q(\sigma) \mid \sigma \in \pi_1(\mathcal{M}) \rangle \subset \mathbb{R}^* \times \mathrm{O}(n)$$

with $Q(\sigma) := (a_\sigma, d\mu_{\sigma^{-1}}^{-1} \circ \mathcal{P}_\sigma^h)$. Here, $\mu_\sigma := \nu_\sigma(u, v, \cdot)$ and $\mathcal{P}_\sigma^h$ is the parallel transport w.r.t. h along some curve in $\mathcal{N}$ from p to $\mu_{\sigma^{-1}}(p)$.

In Section 1.2 we will introduce the screen bundle and screen distributions and explain in detail how these are related to the holonomy and geometry of the underlying Lorentzian manifold. Moreover we will give some first partial answers to Problem B. For example, if on a leaf $\mathcal{L}^\perp$ of the parallel subbundle $\mathbb{L}^\perp$ there exists an *involutive* and *horizontal* screen distribution $\mathbb{S}$ ($\mathbb{S}$ is called horizontal if and only if $[\Gamma(\mathbb{L}), \Gamma(\mathbb{S})] \subset \Gamma(\mathbb{S})$), we obtain

5 We define $\mathbb{R}^* := \mathbb{R} \backslash \{0\}$.

Proposition IV. *Let* $(\mathcal{M}^{(n+2)}, g)$ *be a Lorentzian manifold with special holonomy and* V *a global nowhere vanishing light-like vector field spanning* $\mathbb{L}$*. Assume that, along* $\mathcal{L}^{\perp}$*, there exists a horizontal and involutive realization* $\mathbb{S}$ *of the screen bundle. Then, the potential* $\mathfrak{P}$ *defined throught* $\mathfrak{P} := \nabla^g|_{\mathcal{L}^{\perp}} - \nabla^h$ *takes only values in* $\mathbb{L}$ *and the recurrent vector field* V *on* $\mathcal{M}$ *is parallel w.r.t. the metric* $h = g^R|_{\mathcal{L}^{\perp}}$ *on* $\mathcal{L}^{\perp}$*. In particular this implies that*

$$[R^g(X,Y) - R^h(X,Y)]W \in \Gamma(\mathbb{V}) \quad \textit{for all } X, Y, W \in \Gamma(\mathbb{L}^{\perp})$$

and for the Ricci curvature $\mathrm{Ric}^g|_{\mathbb{L}^{\perp} \times \mathbb{L}^{\perp}} = \mathrm{Ric}^h$.

The subsequent Section 1.3 will be devoted to foliation theory in general and its relations to Lorentzian manifolds with special holonomy in detail. In particular we state a well-known but remarkable result about the connection between geodesic completeness of a foliated Riemannian manifold and the completeness of a leaf with its induced metric, cf. Proposition 1.26, and introduce cohomology theories (i.e. the *basic* and *twisted* cohomology) related to a given foliation. These cohomology groups in turn will be useful to tackle Problem F. The last Section 1.4 provides tools to study Problems D and B by examine the structure of certain covers of Lorentzian manifolds with special holonomy using the fact that a time-orientable Lorentzian manifold $(\mathcal{M}^{(n+2)}, g)$ admits a section $V \in \Gamma(\mathbb{L})$ with $\nabla^g V = \beta \otimes V$. If, moreover, $\ker \beta = \mathbb{L}^{\perp}$, then $(\mathcal{M}^{(n+2)}, g)$ is called *decent*. For decent Lorentzian manifolds we obtain

Proposition V. *Let* $(\mathcal{M}^{(n+2)}, g)$ *be a decent Lorentzian manifold and* $\mathcal{L}^{\perp}$ *a leaf of* $\mathbb{L}^{\perp}$*. Assume that, along* $\mathcal{L}^{\perp}$*, there exists a horizontal and involutive screen distribution* $\mathbb{S}$*. Then the universal cover is diffeomorphic to* $\mathbb{R}^2 \times \widetilde{\mathcal{S}}$*, where* $\widetilde{\mathcal{S}}$ *is the universal cover of a leaf* $\mathcal{S}$ *of* $\mathbb{S}$*. If* $\mathcal{M}$ *is compact, then it is even covered by* $\mathbb{R}^2 \times \mathcal{S}$*.*

Within Chapter 2 we study Lorentzian manifolds with Abelian holonomy which turns out to be equivalent for the manifold to be a pp-wave, see Proposition 2.2, and with the obtained results we solve the Problems D and C for the case of compact pp-waves. Moreover, these results lead to a partial answer to the Ehlers-Kundt Problem (Problem G), i.e. in the compact case. Indeed, as a first step we obtain in Section 2.2 the following two results.

Theorem VI. *For a compact pp-wave the maximal geodesics along the leaves of the parallel distribution* $\mathbb{L}^{\perp}$ *are defined on* $\mathbb{R}$*.*

Theorem VII. *Let* $(\mathcal{M}^{(n+2)}, g)$ *be a pp-wave with parallel light-like vector field* $V \in \Gamma(T\mathcal{M})$ *satisfying the following completeness assumptions:*

(i) The maximal geodesics along the leaves of $\mathbb{L}^{\perp}$ *are defined on* $\mathbb{R}$ *and*

(ii) there exists a complete screen vector field Z*.*

Then, the universal cover $\widetilde{\mathcal{M}}$ *of* $\mathcal{M}$ *is diffeomorphic to* $\mathbb{R}^{n+2}$*. Moreover, the universal cover* $(\widetilde{\mathcal{M}}, \widetilde{g})$ *is globally isometric to a standard pp-wave*

$$\left(\mathbb{R}^{n+2}, \ g^H = 2dudv + 2H(u, x^1, \dots, x^n)du^2 + \delta_{ij}dx^i dx^j\right).$$

Under this isometry, the lift of the parallel vector field V *is mapped to* ∂_v*.*

Combining Theorem VII & VI we finally obtain the following description of the universal cover solving Problem D for compact pp-waves.

Theorem VIII. *The universal cover of an $(n+2)$-dimensional compact pp-wave is globally isometric to a standard pp-wave*

$$\left(\mathbb{R}^{n+2},\ g^H = 2dudv + 2H(u, x^1, \dots, x^n)du^2 + \delta_{ij}dx^i dx^j\right).$$

Under this isometry, the lift of the parallel light-like vector field is mapped to ∂_v.

Using this description we can investigate in Section 2.3 geodesic completeness of compact pp-waves by applying existing results for geodesic completeness of non-compact Lorentzian manifolds with parallel light-like vector field. This answers Problem C w.r.t. compact pp-waves in the context of geodesic completeness.

Theorem IX. *Every compact pp-wave is geodesically complete.*

Again by applying Theorem VIII we study plane waves in Section 2.4 and we obtain as a corollary a solution to the Ehlers-Kundt problem in the compact case.

Corollary X. *Every compact Ricci-flat pp-wave is a plane wave.*

As we have already mentioned earlier, an example for plane waves are Lorentzian manifolds with essentially parallel Weyl tensor W^g, i.e. with $\nabla^g W^g = 0$, but neither $W^g = 0$ nor $\nabla^g R^g = 0$. An interesting property of plane waves is to be (locally) homogeneous, see e.g. [BO03] and in this spirit we study in Section 2.5 the isometry group of Lorentzian manifolds with essentially parallel Weyl tensor and prove the following.

Theorem XI. *Let $(\widetilde{\mathcal{M}}^{(n+2)}, \widetilde{g})$ denote a Lorentzian manifold with essentially parallel Weyl tensor as in Proposition 2.32. Then the identity component $\mathrm{Isom}^0(\widetilde{\mathcal{M}}^{(n+2)}, \widetilde{g})$ of the isometry group of $(\widetilde{\mathcal{M}}^{(n+2)}, \widetilde{g})$ is isomorphic to $\mathcal{S} \ltimes \mathrm{He}(n)$, where $\mathcal{S} \subset \mathrm{SO}(n)$ is a connected Lie subgroup of $\mathrm{SO}(n)$ with Lie algebra $\mathfrak{s} := \mathrm{span}\{F \in \mathfrak{so}(n) \mid [A, F] = 0\}$ which is non-trivial if and only if $A \in \mathrm{End}(V)$ has at least one eigenspace of dimension greater than one.*

In particular we thus obtain an answer to a question of Derdziński posed in [DR07] in the Lorentzian case, namely:

Corollary XII. *A compact Lorentzian manifold with essentially parallel Weyl tensor cannot be locally homogeneous.*

In Chapter 3 we will tackle Problem F by using results in [Lär11], Proposition IV together with foliation theory and basic resp. twisted cohomology to obtain the following result which is in some sense a "Lorentzian version" of the classical Bochner result in the Riemannian case.

Theorem XIII. *Let $(\mathcal{M}^{(n+2)}, g)$ be an orientable $(n+2)$-dimensional decent Lorentzian manifold. Assume that the leaves of the codimension one foliation induced by the distribution $\mathbb{L}^\perp$ are compact and $\mathrm{Ric}|_{\mathbb{L}^\perp \times \mathbb{L}^\perp} \geq 0$.*

(i) If $\mathcal{M}$ is compact, then $b_1(\mathcal{M}) \leq n+2$ and $b_1(\mathcal{M}) = n+2$ if and only if $\mathcal{M}$ is – up to finite cover – diffeomorphic (homeomorphic if $\dim \mathcal{M} = 4$*) to the torus and g has light-like hypersurface curvature.*[6]

(ii) If $\mathcal{M}$ is non-compact, then $b_1(\mathcal{M}) \leq n+1$ and $b_1(\mathcal{M}) = n+1$ if and only if $\mathcal{M}$ is isometric to $\mathbb{R} \times \mathbb{T}^{n+1}$ and g has light-like hypersurface curvature.

In both cases, the leaves of $\mathbb{L}^\perp$ are all diffeomorphic to the torus $\mathbb{T}^{n+1}$.

Last but not least, we use in Chapter 4 a construction of Lorentzian metrics on total spaces $\mathcal{M}$ of circle bundles $\pi : \mathcal{M} \longrightarrow \mathcal{N}$ over a Riemannian manifold $(\mathcal{N}, h)$ with prescribed first Chern class $c_1(\mathcal{M}) = \omega$ for some $\omega \in H^2(\mathcal{N}, \mathbb{Z})$ which has already been studied in [Lär11] with a similar motivation to ours. We will examine it in order to provide partial answers and examples to Problems D & E and finally to Problem A. For this purpose we start in Section 4.1 with basic computations for these metrics which are defined as follows. Take any closed 2-form $\Psi \in \Omega^2(\mathcal{N})$ such that Ψ represents ω in the de Rham cohomology and a corresponding connection $A \in \Omega^1(\mathcal{M}, \mathrm{i}\mathbb{R})$ with curvature $F^A = dA = -2\pi\mathrm{i}\pi^*\Psi$. Then we define a Lorentzian metric on $\mathcal{M}$ by

$$g := 2\mathrm{i}A \odot \pi^*\eta + f \cdot \pi^*\eta \odot \pi^*\eta + \pi^*h$$

for a nowhere vanishing closed 1-form $\eta \in \Omega^1(\mathcal{N})$ and a smooth function $f \in C^\infty(\mathcal{M})$. We refer to this construction by saying that *$(\mathcal{M}^{(n+2)}, g)$ is of type (Ψ, A, η, f) over $(\mathcal{N}, h)$*. In Section 4.2 we continue the investigation of these manifolds under the viewpoint of completeness and obtain, together with the results in [Lär11], complete examples with non-trivial topology and holonomy of type 2, related to Problem E. By computing the curvature of $(\mathcal{M}^{(n+2)}, g)$ in Section 4.3 we investigate whether the Lorentzian manifolds of type (Ψ, A, η, f) over $(\mathcal{N}, h)$ produce examples for Lorentzian Einstein manifolds. As it turns out, this is only the case if the cosmological constant is zero, i.e. if $(\mathcal{M}^{(n+2)}, g)$ is Ricci-flat. Hence we arrive at:

Theorem XIV. *Let $\mathcal{N} := \mathcal{B} \times \mathbb{S}^1$ or $\mathcal{N} := \mathcal{B} \times \mathbb{R}$ with $h := h_\mathcal{B} \oplus du^2$ for an n-dimensional Riemannian manifold $(\mathcal{B}, h_\mathcal{B})$. Moreover, let $(\mathcal{B}, h_\mathcal{B})$ be Ricci-flat and $\eta := du$ the coordinate 1-form on $\mathbb{S}^1$ resp. $\mathbb{R}$. Choose $\omega \in H^1_{\mathrm{dR}}(\mathcal{B}) \cap H^1(\mathcal{B}, \mathbb{Z})$ and a representative $\alpha \in \omega$ and consider the $\mathbb{S}^1$-bundle $\pi : \mathcal{M} \longrightarrow \mathcal{N}$ with $c_1(\mathcal{M}) = [\alpha \wedge \eta]$. Finally, choose $\Psi := \alpha \wedge \eta$ and $f := \widehat{f} \circ \pi \in C^\infty(\mathcal{M})$, where $\widehat{f} := f_\mathcal{B} \cdot f_{\mathbb{S}^1}$ with $f_\mathcal{B} \in C^\infty(\mathcal{B})$ and $f_{\mathbb{S}^1} \in C^\infty(\mathbb{S}^1)$.*

Then, the Lorentzian manifold $(\mathcal{M}^{(n+2)}, g)$ of type (Ψ, A, η, f) over $(\mathcal{N}, h)$ is Ricci-flat if and only if $\Delta_{h_\mathcal{B}}(f_\mathcal{B}) = -4\,\mathrm{div}_{h_\mathcal{B}}(\alpha)$.

The latter theorem thus provides a contribution to Problem A and by applying the results obtained in Section 4.2 we additionally obtain completeness of the examples.

Corollary XV. *Every compact Ricci-flat Lorentzian manifold occurring in Corollary 4.13 is complete. This even holds for arbitrary $f_\mathcal{B} \in C^\infty(\mathcal{B})$.*

Hence, the presented circle bundle construction produces compact, complete Ricci-flat Lorentzian manifolds with non-trivial topology. What remains under the viewpoint

6 A decent Lorentzian manifold $(\mathcal{M}, g)$ is said to have *light-like hypersurface curvature*, if and only if the curvature R^g satisfies $R^g(X,Y)W \in \Gamma(\mathbb{L})$ for all $X, Y, W \in \Gamma(\mathbb{L}^\perp)$.

of Problem A is the question about the holonomy of the examples coming from Theorem XIV. Concerning this question we find in Section 4.4, by investigating the universal cover $\widetilde{\mathcal{M}} \simeq \mathbb{R}^2 \times \mathcal{S}$ of the Lorentzian manifolds of type (Ψ, A, η, f) over $(\mathcal{N}, h)$ that

Theorem XVI. *Let $(\mathcal{M}^{(n+2)}, g)$ be a Lorentzian manifold of type (Ψ, A, η, f) over $(\mathcal{N}, h)$ with the data chosen as in Theorem XIV and $f \in C^\infty(\mathcal{N})$ such that $\mathrm{Hess}_B f|_{\mathcal{B}}$ is non-degenerate in a point. Then the full holonomy group is given by*

$$\mathrm{Hol}_x(\mathcal{M}^{(n+2)}, g) = O \cdot \mathrm{Hol}^0_q(\mathcal{B}, h_{\mathcal{B}}) \ltimes \mathbb{R}^n,$$

where $(\mathrm{pr}_{\mathcal{B}} \circ \pi \circ \Phi)(\widetilde{x}) = q$, $\widetilde{x} = (u, v, p)$, $\Phi(\widetilde{x}) = x$ and

$$O := \left\langle (d\mu_{\sigma^{-1}})^{-1} \circ \mathcal{P}^{\Theta}_{\sigma} \mid \sigma \in \pi_1(\mathcal{M}) \right\rangle \subset \mathrm{O}(n),$$

with the notations as in Theorem III. Moreover, we can replace $\pi_1(\mathcal{M})$ by $\pi_1(\mathcal{B})$ in O, if $\pi_1(\mathcal{B})$ is split.[7] In this case we actually have

$$\mathrm{Hol}_x(\mathcal{M}^{(n+2)}, g) = \mathrm{Hol}_q(\mathcal{B}, h_{\mathcal{B}}) \ltimes \mathbb{R}^n.$$

As a consequence we can – by using Theorem XIV together with Theorem XVI – produce (complete) examples for Ricci-flat Lorentzian manifolds with prescribed holonomy and thus arrive at a contribution to Problem A.

Finally, in light of Problem E, we provide in the last Section 4.4 examples for complete Lorentzian manifolds $(\mathcal{M}^{(n+2)}, g)$ with holonomy of type 4 and non-trivial topology, i.e. such that at least the "direction" of the parallel light-like vector field does not split globally from $\mathcal{M}$. Indeed, the constructed examples are diffeomorphic to $\mathbb{S}^1 \times \mathcal{P} \times \mathbb{R}^m$, where the parallel light-like vector field is a vector field on $\mathcal{P}$ and $\pi : \mathcal{P} \longrightarrow \mathbb{T}^k$ is a non-trivial circle bundle over the k-torus. The obtained examples can be summarized as follows.

Theorem XVII. *For each Abelian Riemannian holonomy algebra $\mathfrak{g} \subset \mathfrak{so}(k)$ there exists a complete indecomposable but non-irreducible Lorentzian manifold with holonomy of type 4 possessing $\mathfrak{g}$ as orthogonal part.*

7 This is a technical definition which can be found in Section 4.4 as Definition 4.17 on page 78.

Acknowledgement

As a student of the "Baumschule"[8] I would like to express my gratitude to my supervisor *Prof. Helga Baum*, for her support and imperturbable encouragement over the past years throughout my studies. I am also grateful to many other people, above all *Thomas Leistner* for many helpful discussions, proofreading of papers, fruitful ideas leading to our joint work and for inviting me to Adelaide in 2012, but also to *Peter Schemel* and *Christoph Stadtmüller* for many helpful discussions. Beside those, I would like to especially thank my *family*, *parents-* and *grandparents-in-law* for supporting me and my girlfriend Sina in all areas after the birth of our daughter Pia. In particular I am grateful to and proud of my grandparents *Hans* and *Ilse* which took care of Pia while we had no place in a Kindergarten for her.

Finally I would like to thank the *Berlin Mathematical School (BMS)* for providing me a scholarship which laid the foundations for the preparation of this thesis.

Danksagung

Als Student der "Baumschule"[9] möchte ich meinen großen Dank meiner Betreuerin *Prof. Helga Baum* ausdrücken. Nur durch ihr enormes Engagement in der Lehre und Unterstützung bei allen möglichen Fragen konnte diese Doktorarbeit entstehen. Ich danke auch vielen weiteren Personen, allen voran *Thomas Leistner* für viele hilfreiche Diskussionen, Korrekturlesen von Papern, viele gute Ideen, die letztlich zu unserer gemeinsamen Arbeit geführt haben, sowie dafür, dass ich ihn 2012 in Adelaide besuchen durfte. Außerdem danke ich *Peter Schemel* und *Christoph Stadtmüller* für viele hilfreiche Diskussionen.

Neben diesen Personen aus meiner mathematischen Welt möchte ich besonders auch meiner *Familie*, *Schwiegereltern* und *-großeltern* für die Unterstützung danken, die meine Freundin Sina und ich während der Zeit unserer Promotion und speziell nach der Geburt unserer Tochter Pia erhalten haben. Besonders dankbar und stolz bin ich meinen Großeltern *Ilse* und *Hans* gegenüber, welche Pia während der Zeit betreut haben als wir noch keinen Kindergartenplatz gefunden hatten. Hierdurch war es mir möglich, weiter an meiner Promotion zu arbeiten und es ist damit auch ihr Verdienst, dass diese nun fertig gestellt wurde.

Zuletzt möchte ich auch besonderen Dank der *Berlin Mathematical School (BMS)* aussprechen, die mir ein Stipendium gewährt und damit diese Arbeit erst möglich gemacht hat.

8 Participants of the conference "Lorentzian and Conformal Geometry" which took place in Greifswald in March 2014 will understand this twofold indication.

9 Teilnehmer der Konferenz "Lorentzian and Conformal Geometry" im März 2014 in Greifswald werden die Zweideutigkeit verstehen.

1 PRELIMINARIES

1.1 LORENTZIAN HOLONOMY GROUPS

Within this section we will explain essentially the facts concerning holonomy theory required to understand the results of this thesis. Basically, this includes the classical results in the Riemannian case and the more recent results for Lorentzian manifolds.[1] However, more comprehensive overview articles about holonomy theory of semi-Riemannian[2] manifolds can be found e.g. in [LG08, Bau12].

General Facts

A central object of this thesis is the holonomy group of a semi-Riemannian geometric vector bundle $(E, \nabla^E, \langle\cdot,\cdot\rangle)$ over a manifold $\mathcal{M}^{(n+2)}$ and in particular the holonomy of semi-Riemannian manifolds $(\mathcal{M}^{(n+2)}, g)$.

Definition 1.1. *Let $(E, \nabla^E, \langle\cdot,\cdot\rangle)$ be a geometric vector bundle of rank $k \in \mathbb{N}$ over a manifold $\mathcal{M}^{(n+2)}$ with semi-Riemannian bundle metric $\langle\cdot,\cdot\rangle$ of signature (p,q) with $p+q=k$. We denote with*

$$\mathrm{Hol}_x(E, \nabla^E) := \{\mathcal{P}^E_\gamma \mid \gamma \in \Omega(x)\} \subset \mathrm{O}(E_x, \langle\cdot,\cdot\rangle_x) \simeq \mathrm{O}(p,q)$$

the **holonomy group** *of $(E, \nabla^E, \langle\cdot,\cdot\rangle)$ in $x \in \mathcal{M}$ and by*

$$\mathrm{Hol}^0_x(E, \nabla^E) := \{\mathcal{P}^E_\gamma \mid \gamma \in \Omega_0(x)\} \subset \mathrm{O}(E_x, \langle\cdot,\cdot\rangle_x) \simeq \mathrm{O}(p,q)$$

the **reduced holonomy group** *of $(E, \nabla^E, \langle\cdot,\cdot\rangle)$ in $x \in \mathcal{M}$. Here, $\mathcal{P}^E_\gamma$ denotes the parallel displacement along γ w.r.t. the connection of ∇^E, $\Omega(x)$ denotes the set of piecewise smooth curves closed in $x \in \mathcal{M}$ and $\Omega_0(x)$ the subset of curves in $\Omega(x)$ which are null-homotopic.*

For a semi-Riemannian manifold $(\mathcal{M}^{(n+2)}, g)$ we denote with $\mathrm{Hol}^{(0)}_x(\mathcal{M}^{(n+2)}, g)$ the (reduced) holonomy of $(T\mathcal{M}, \nabla^g, g)$, where ∇^g denotes the Levi-Civita connection of g.

If $y \in \mathcal{M}$ is another point, then the holonomy groups in x and y are conjugated:

$$\mathrm{Hol}_y(\mathcal{M}, g) = \mathcal{P}^g_\sigma \circ \mathrm{Hol}_x(\mathcal{M}, g) \circ (\mathcal{P}^g_\sigma)^{-1}$$

with σ denoting a smooth curve connecting x and y. Hence it makes sense to talk about *the* holonomy group of $(\mathcal{M}, g)$, omitting the point $x \in \mathcal{M}$.

1 Within this thesis, all manifolds are assumed to be smooth, connected and without boundary.
2 We call a metric *semi-Riemannian* if it has arbitrary signature (p,q) where p is the number of -1 and q the number of $+1$ in its normal form. We say that it is *Riemannian*, if it has signature $(0,q)$ and *Lorentzian* if it has signature $(1,q)$.

The reduced holonomy group $\mathrm{Hol}^0_x(\mathcal{M}^{(n+2)}, g)$ is the connected component of the identity in $\mathrm{Hol}_x(\mathcal{M}^{(n+2)}, g)$ [Bes87, 10.48]. Moreover, the reduced holonomy group is a normal subgroup in the whole holonomy group and for all $x \in \mathcal{M}$ the map

$$\pi_1(\mathcal{M}, x) \ni [\gamma] \longmapsto [\mathcal{P}^g_\gamma] \in \mathrm{Hol}_x(\mathcal{M}, g) / \mathrm{Hol}^0_x(\mathcal{M}, g) \tag{1.1}$$

surjects $\pi_1(\mathcal{M}, x)$ homomorphically onto $H_x := \mathrm{Hol}_x(\mathcal{M}, g) / \mathrm{Hol}^0_x(\mathcal{M}, g)$ [Bes87, 10.15]. In particular, the quotient group H_x is at most countable as so is $\pi_1(\mathcal{M}, x)$. If $(\widetilde{\mathcal{M}}, \widetilde{g})$ is the universal cover of $(\mathcal{M}^{(n+2)}, g)$ with $\pi : \widetilde{\mathcal{M}} \longrightarrow \mathcal{M}$ denoting the covering map, then every null-homotopic loop in $\mathcal{M}$ lifts to a null-homotopic loop in $\widetilde{\mathcal{M}}$. Hence, for every $\widetilde{x} \in \widetilde{\mathcal{M}}$:

$$\mathrm{Hol}^0_{\pi(\widetilde{x})}(\mathcal{M}^{(n+2)}, g) \cong \mathrm{Hol}_{\widetilde{x}}(\widetilde{\mathcal{M}}^{(n+2)}, \widetilde{g}). \tag{1.2}$$

With this fact in mind we see that the following recent result [BLL14, Proposition 3] generalizes (1.1) appropriately.

Proposition 1.2. *Let $(\mathcal{M}, g)$ be a semi-Riemannian manifold and $\pi : (\widehat{\mathcal{M}}, \widehat{g}) \longrightarrow (\mathcal{M}, g)$ a semi-Riemannian covering map. Then, for any $x \in \mathcal{M}$ and $\widehat{x} \in \pi^{-1}(x)$ we have:*

(i) For any loop $\widehat{\gamma}$ at $\widehat{x}$, the homomorphism $\iota : \mathcal{P}^{\widehat{g}}_{\widehat{\gamma}} \longmapsto \mathcal{P}^g_{\pi\circ\widehat{\gamma}}$ is injective and the image of $\mathrm{Hol}_{\widehat{x}}(\widehat{\mathcal{M}}, \widehat{g})$ under ι is normal in $\mathrm{Hol}_x(\mathcal{M}, g)$.

(ii) $\Pi : \pi_1(\mathcal{M}) \ni \sigma \longmapsto [\mathcal{P}_\gamma] \in \mathrm{Hol}_x(\mathcal{M}, g) / \mathrm{Hol}_{\widehat{x}}(\widehat{\mathcal{M}}, \widehat{g})$ is a surjective homomorphism, where σ is interpreted as an element of the isometry group of $(\widehat{\mathcal{M}}, \widehat{g})$ and γ is a loop at x whose lift starts at $\widehat{x}$ and ends in $\sigma^{-1}(\widehat{x})$.

(iii) For any loop γ at x it holds $\mathcal{P}^g_\gamma = d\sigma_{\sigma^{-1}(\widehat{x})} \circ \mathcal{P}^{\widehat{g}}_{\widehat{\gamma}}$, where $\widehat{\gamma}$ is the lift of γ starting at $\widehat{x}$ and ending in $\sigma^{-1}(\widehat{x})$ for $\sigma \in \pi_1(\mathcal{M})$. In particular, $Q(\sigma) := (d\sigma^{-1}|_x)^{-1} \circ \mathcal{P}^{\widehat{g}}_{\widehat{\gamma}}$ is a representative of $\Pi(\sigma)$ and hence

$$\mathrm{Hol}_x(\mathcal{M}, g) = Q \cdot \iota(\mathrm{Hol}_{\widehat{x}}(\widehat{\mathcal{M}}, \widehat{g}))$$

with Q denoting the group generated by the $Q(\sigma)$, $\sigma \in \pi_1(\mathcal{M})$.

Finally, we point out that, given a semi-Riemannian product $(\mathcal{M}, g) = (\mathcal{M}_1, g_1) \times (\mathcal{M}_2, g_2)$, the holonomy of $(\mathcal{M}, g)$ is simply the product of the holonomy of $(\mathcal{M}_1, g_1)$ and $(\mathcal{M}_2, g_2)$, [Bes87, 10.35], i.e.

$$\mathrm{Hol}_{(x_1,x_2)}(\mathcal{M}, g) = \mathrm{Hol}_{x_1}(\mathcal{M}_1, g_1) \times \mathrm{Hol}_{x_2}(\mathcal{M}_2, g_2). \tag{1.3}$$

Being a Lie group, the holonomy group of a semi-Riemannian manifold has a corresponding Lie algebra, the *holonomy algbra of* $\mathfrak{hol}_x(\mathcal{M}, g)$, denoted $\mathfrak{hol}_x(\mathcal{M}, g)$. The well-known *Holonomy Theorem of Ambrose and Singer* states that the holonomy algebra can be computed as

$$\mathfrak{hol}_x(\mathcal{M}, g) = \mathrm{span}\{(\mathcal{P}^g_\gamma)^{-1} \circ R^g(\mathcal{P}^g_\gamma(v), \mathcal{P}^g_\gamma(w)) \circ \mathcal{P}^g_\gamma \mid v, w \in T_x\mathcal{M},\ \gamma(0) = x\}, \tag{1.4}$$

with R^g denoting the Riemannian curvature tensor.[3] This already gives an idea why it is often less complicated to compute the reduced instead of the full holonomy group of

3 For the curvature tensor we use throughout this thesis the convention $R^g(X, Y)Z := \nabla^g_X \nabla^g_Y Z - \nabla^g_Y \nabla^g_X Z - \nabla^g_{[X,Y]} Z$.

a manifold. However, we will tackle both problems for various Lorentzian manifolds within this thesis.

One of the most important reasons why mathematicians are interested in holonomy of semi-Riemannian manifolds is their close relation to parallel tensors on the manifolds. This is the essence of the so called *holonomy principle* which can be summarized as the following theorem.

Theorem 1.3. *Let $(\mathcal{M}, g)$ be a semi-Riemannian manifold. For any tensor bundle $\mathscr{T}$ on $(\mathcal{M}, g)$ there is the bijective correspondence*

$$\left\{ \mathrm{t} \in \mathscr{T}_x \mid \mathrm{Hol}_x(\mathcal{M}, g)\mathrm{t}_x = \mathrm{t}_x \right\} \longleftrightarrow \left\{ T \in \mathscr{T} \mid \nabla^{\mathscr{T}} T = 0 \right\}$$

with $\nabla^{\mathscr{T}}$ denoting the connection induced on $\mathscr{T}$ via the Levi-Civita connection of g.

According to this, the existence of holonomy invariant subspaces $E \subset T_x\mathcal{M}$ of the tangent space seem to be of high importance. Consequently, the natural representation of $\mathrm{Hol}_x(\mathcal{M}, g)$ on $\mathrm{O}(T_x\mathcal{M})$, denoted by

$$\rho : \mathrm{Hol}_x(\mathcal{M}, g) \longrightarrow \mathrm{O}(T_x\mathcal{M}, g_x),$$

is named *irreducible*, if there exists no proper holonomy invariant subspace $E \subset T_x\mathcal{M}$ and *weakly-irreducible* if there no proper non-degenerate holonomy invariant subspace. For short we say that a semi-Riemannian manifold is (weakly-)irreducible, if so is its holonomy representation ρ. It is also common to call weakly-irreducible manifolds *indecomposable* meaning that their holonomy representation does not decompose into non-degenerate sub-representations.[4] Obviously, every indecomposable Riemannian manifold is also irreducible. However, for semi-Riemannian manifolds which are not Riemannian, this is in general not correct and makes it more difficult to classify their holonomy groups.

Nevertheless, the starting point in the classification of holonomy groups is the de Rham/Wu decomposition theorem [DR52, Wu64].

Theorem 1.4. *Any simply-connected, geodesically complete semi-Riemannian manifold is globally isometric to a product of a flat manifold (which is possibly zero dimensional) and indecomposable non-flat manifolds.*

As a result, an indecomposable semi-Riemannian manifold cannot, even locally, decompose and due to (1.3) the problem of classifying holonomy representations reduces to the case of classifying the indecomposable semi-Riemannian manifolds. For the irreducible case, the possible holonomy groups are well-known. For symmetric spaces one can consult e.g. [Bes87, Section 10.G] or [Ber57]. For the non-symmetric cases these were classified by Berger [Ber55]. His list contains the possible irreducible holonomy groups of simply-connected, not locally-symmetric semi-Riemannian manifolds and is commonly referred to as *Berger's list*.

4 Note that for every holonomy invariant subspace E, its orthogonal complement $E^{\perp}$ is holonomy invariant, too. If the metric is non-degenerate on E it so is on $E^{\perp}$ and we obtain a decomposition $T_xM = E \oplus E^{\perp}$. Hence ρ decomposes into two sub-representations in this case.

Riemannian and Lorentzian Holonomy

At this point we begin to restrict our introduction into holonomy theory to the cases of Riemannian and Lorentzian geometries since these are studied in this thesis. In the Riemannian case, the above mentioned Berger list contains only the groups $\mathrm{SO}(n)$, $\mathrm{U}(\frac{n}{2})$, $\mathrm{SU}(\frac{n}{2})$, $\mathrm{Sp}(\frac{n}{4})$, $\mathrm{Sp}(\frac{n}{4}) \cdot \mathrm{Sp}(1)$, G_2 and $\mathrm{Spin}(7)$, where n is the dimension of the manifold. In view of Chapter 4, we stress that the only Riemannian holonomy groups of this list *not* implying vanishing Ricci curvature are $\mathrm{SO}(n)$, $\mathrm{U}(\frac{n}{2})$ (Kähler manifolds) and $\mathrm{Sp}(\frac{n}{4}) \cdot \mathrm{Sp}(1)$ (quaternionic Kähler manifolds). As we will see in the next paragraph, the Riemannian holonomy groups do also play an important role within the classification of the indecomposable but non-irreducible Lorentzian holonomy groups.

If we take a closer look into Berger's list, it is conspicuous that, except for $\mathrm{SO}^0(1, n+1)$, it does not contain a subgroup of the Lorentz group $\mathrm{O}(1, n+1)$. Actually, this is a special property in Lorentzian signature and a consequence of the following fact [DSO01].

Theorem 1.5. *Any connected Lie subgroup of* $\mathrm{O}(1, n+1)$ *acting irreducibly on* $\mathbb{R}^{1,n+1}$ *is equal to* $\mathrm{SO}^0(1, n+1)$.

This theorem motivates to make the following definition clarifying the title of this thesis.

Definition 1.6. *A Lorentzian manifold whose holonomy representation acts indecomposably but non-irreducibly is said to have* **special holonomy**.

Due to Theorem 1.5 every Lorentzian manifold whose reduced holonomy group not equals $\mathrm{SO}^0(1, n+1)$ has to have special holonomy. Consequently, its holonomy representation ρ needs to preserve a degenerate subspace $W \subset T_x\mathcal{M}$ and hence a holonomy invariant light-like line $L := W \cap W^{\perp_g}$. By the holonomy principle (Theorem 1.3), the holonomy group $\mathrm{Hol}_x(\mathcal{M}^{(n+2)}, g)$ is contained in the stabilizer $\mathrm{O}(T_x\mathcal{M}, g_x)_L$ of L in $\mathrm{O}(T_x\mathcal{M}, g_x)$. To describe the stabilizer in more detail, let $v, e_1, \dots, e_n, w$ be a basis of $(T_x\mathcal{M}, g_x)$ such that g_x takes the form

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & \mathbb{I}_n & 0 \\ 1 & 0 & 0 \end{pmatrix}, \tag{1.5}$$

with $\mathbb{I}_n$ denoting the $(n \times n)$-identity matrix and such that $L = \mathbb{R}v$. Then

$$\begin{aligned} \mathrm{O}(T_x\mathcal{M}, g_x)_L &= (\mathbb{R}^* \times \mathrm{O}(n)) \ltimes \mathbb{R}^n \\ &= \left\{ \begin{pmatrix} a^{-1} & x^t & -\frac{1}{2}a||x||^2 \\ 0 & A & -aAx \\ 0 & 0 & a \end{pmatrix} \middle| \, a \in \mathbb{R}^*, \ x \in \mathbb{R}^n, \ A \in \mathrm{O}(n) \right\} \end{aligned} \tag{1.6}$$

and for the Lie algebra

$$\mathfrak{so}(T_x\mathcal{M}, g_x)_L = (\mathbb{R} \oplus \mathfrak{so}(n)) \ltimes \mathbb{R}^n$$

$$= \left\{ \begin{pmatrix} a & x^t & 0 \\ 0 & A & -x \\ 0 & 0 & -a \end{pmatrix} \middle| \, a \in \mathbb{R}, \ x \in \mathbb{R}^n, \ A \in \mathfrak{so}(n) \right\} \tag{1.7}$$

w.r.t. a basis such that g_x takes the form (1.5). As a first step, Bérard-Bergery and Ikemakhen determined the possible algebraic types of weakly-irreducible subalgebras of $\mathfrak{so}(T_x\mathcal{M}, g_x)_L$, cf. [BBI93].

Theorem 1.7. *Let $\mathfrak{h} \subset \mathfrak{so}(1, n+1)_L$ be a weakly-irreducible subalgebra and let $\mathfrak{g} := \mathrm{pr}_{\mathfrak{so}(n)}(\mathfrak{h})$ denote the orthogonal part. Then $\mathfrak{h}$ belongs to one of the following types:*

Type 1: $\mathfrak{h} = (\mathbb{R} \oplus \mathfrak{g}) \ltimes \mathbb{R}^n$,

Type 2: $\mathfrak{h} = \mathfrak{g} \ltimes \mathbb{R}^n$,

Type 3: $\mathfrak{h} = \{(\varphi(X), X+Y, z) \mid X \in \mathfrak{z}(\mathfrak{g}), \ Y \in [\mathfrak{g}, \mathfrak{g}], z \in \mathbb{R}^n\}$, where $\varphi : \mathfrak{z}(\mathfrak{g}) \twoheadrightarrow \mathbb{R}$ is a surjective homomorphism,

Type 4: $\mathfrak{h} = \{(0, X+Y, \varphi(X)+z) \mid X \in \mathfrak{z}(\mathfrak{g}), \ Y \in [\mathfrak{g}, \mathfrak{g}], z \in \mathbb{R}^k\}$, where $\mathbb{R}^n = \mathbb{R}^m \oplus \mathbb{R}^k$, $0 < m < n$, $\mathfrak{g} \subset \mathfrak{so}(k)$ and $\varphi : \mathfrak{z}(\mathfrak{g}) \twoheadrightarrow \mathbb{R}^m$ is a surjective homomorphism.

We will say that a *Lorentzian manifold has holonomy of type 1,2,3 or 4*, if its holonomy algebra is of this type.

Due to Theorem 1.7 the classification of Lorentzian holonomy algebras is reduced to the study of the orthogonal parts $\mathfrak{g} = \mathrm{pr}_{\mathfrak{so}(n)}(\mathfrak{hol}(\mathcal{M}^{(n+2)}, g))$. Of course, this is the point were the Riemannian holonomy groups come into play again. Namely, Leistner and Galaev proved the following [Lei07, Gal06].

Theorem 1.8. *Let $H \subset \mathrm{SO}^0(1, n+1)$ be a connected subgroup acting indecomposably but non-irreducibly. Then H is the reduced holonomy group of a Lorentzian manifold if and only if its orthogonal part is a Riemannian holonomy group.*

The orthogonal part $\mathfrak{g}$ can be described differently. Recall that a Lorentzian manifold $(\mathcal{M}^{(n+2)}, g)$ with special holonomy admits a holonomy invariant light-like line L in each point of the manifold. Via parallel translation this gives rise to a parallel line subbundle $\mathbb{L}$ of the tangent bundle and since $L \subset L^\perp$ additionally implies a parallel codimension one subbundle $\mathbb{L}^\perp$. We obtain a filtration

$$\mathbb{L} \subset \mathbb{L}^\perp \subset T\mathcal{M} \tag{1.8}$$

of the tangent bundle. The quotient

$$\Sigma := \mathbb{L}^\perp / \mathbb{L} \tag{1.9}$$

then gives rise to an n-dimensional vector bundle over $\mathcal{M}$ and inherits a connection ∇^Σ from the Levi-Civita connection ∇^g. This is due to the fact that the subbundles $\mathbb{L}$ and $\mathbb{L}^\perp$ are parallel w.r.t. ∇^g since they are holonomy invariant. Hence, for all $X \in \Gamma(T\mathcal{M})$ and $Y \in \Gamma(\mathbb{L}^\perp)$ we define

$$\nabla^\Sigma_X[Y] := [\nabla^g_X Y], \tag{1.10}$$

with $[\cdot] : \mathbb{L}^\perp \twoheadrightarrow \Sigma$. Moreover, Σ can be equipped with a positive definite bundle metric $\langle\cdot,\cdot\rangle_\Sigma$ induced by g:

$$\langle [X],[Y]\rangle_\Sigma := g(X,Y), \tag{1.11}$$

with $X, Y \in \Gamma(\mathbb{L}^\perp)$. The importance of Σ will be intensively discussed within the next section. But at this point we may cite the following alternative description of the orthogonal part of the holonomy group, cf. [Lei06, Corollary 1] and [BLL14, Proposition 2]:

Proposition 1.9. $\mathrm{Hol}_x(\Sigma, \nabla^\Sigma) = \mathrm{pr}_{\mathrm{O}(n)}(\mathrm{Hol}_x(\mathcal{M}^{(n+2)}, g))$.

For Lorentzian manifolds with special holonomy the metric g has a certain local description which is due to Walker [Wal50]

Theorem 1.10. *Let $(\mathcal{M}^{(n+2)}, g)$ be a Lorentzian manifold with special holonomy. Then around any point $x \in \mathcal{M}$ there are coordinates $(\mathcal{U}, (v, x_1, \ldots, x_n, u))$ such that the metric g takes the form*

$$g|_\mathcal{U} = 2dudv + 2Hdu^2 + 2\sum_{i=1}^n A_i dx_i du + \sum_{i,j=1}^n h_{ij} dx_i dx_j \tag{1.12}$$

with $A_i, h_{ij} \in C^\infty(\mathcal{U})$ s.t. $\frac{\partial A_i}{\partial v} = \frac{\partial h_{ij}}{\partial v} = 0$ and $H \in C^\infty(\mathcal{U})$. In this coordinates the parallel light-like line is spanned by ∂_v.

We end this section with examples for Lorentzian manifolds with special holonomy of type 1 and type 2. Of course, for the types 3 and 4 only few examples exist [Gal06, Baz09, Lei06] and in Chapter 4 we provide some new examples at least for type 4. However, to construct Lorentzian manifolds with holonomy of type 1 or 2 one can use the following construction principle, cf. [Lei02], [BLL14, Proposition 4].

Theorem 1.11. *Let $\mathcal{M} := \mathbb{R}^2 \times \mathcal{N}$ and $g = 2dudv + 2Hdu^2 + h$ with $(\mathcal{N}, h)$ a n-dimensional Riemannian manifold. Then $(\mathcal{M}, g)$ has full holonomy*

$$\mathrm{Hol}(\mathcal{M}, g) = \begin{cases} \mathrm{Hol}(\mathcal{N}, h) \ltimes \mathbb{R}^n, & \partial_v H = 0, \\ (\mathbb{R}^+ \times \mathrm{Hol}(\mathcal{N}, h)) \ltimes \mathbb{R}^n, & \partial_v H \neq 0. \end{cases}$$

In particular $(\mathcal{M}, g)$ is time-orientable[5].

More generally, consider $\mathbb{R}^2 \times \mathcal{N}$ equipped with a metric $\widetilde{g}$ of the form

$$\widetilde{g}_{(u,v,p)} = 2dudv + H(u,p)du^2 + A_u \odot du + h_p \tag{1.13}$$

with $A = \{A_u\}$ a family of one-forms on $\mathcal{N}$ and $(\mathcal{N}, h)$ a simply-connected Riemannian manifold. In order to describe the full holonomy of Lorentzian manifolds $(\mathcal{M}^{(n+2)}, g)$ whose universal cover is isometric to $\mathbb{R}^2 \times \mathcal{N}$ with a metric of the form (1.13) we can use the following adaption of Proposition 1.2.

5 A Lorentzian manifold $(\mathcal{M}, g)$ is said to be time-orientable if and only if it admits a nowhere vanishing timelike vector field X, i.e. with $g(X, X) < 0$.

Theorem 1.12 ([BLL14, Theorem 3]). *Let $(\mathcal{M}^{(n+2)}, g)$ be a Lorentzian manifold with special holonomy universally covered by $(\widetilde{\mathcal{M}}, \widetilde{g})$ with $\widetilde{\mathcal{M}} = \mathbb{R}^2 \times \mathcal{N}$ and $\widetilde{g}$ as in (1.13). Then each isometry σ of $(\widetilde{\mathcal{M}}, \widetilde{g})$ is of the form*[6]

$$\sigma(u, v, p) = (a_\sigma^{-1} u + b_\sigma, a_\sigma v + \tau_\sigma(u, v, p), \nu_\sigma(u, v, p)) \tag{1.14}$$

with $a_\sigma \in \mathbb{R}^*, b_\sigma \in \mathbb{R}, \tau_\sigma \in C^\infty(\widetilde{\mathcal{M}})$ with $\partial_v(\tau_\sigma) = 0$ and smooth $\nu_\sigma : \widetilde{\mathcal{M}} \longrightarrow \mathcal{N}$ such that $\partial_v(\nu_\sigma) = 0$ and $\nu(u, v, \cdot)$ is an isometry of $(\mathcal{N}, h)$ for all $u, v \in \mathbb{R}$. Then we find

$$\mathrm{Hol}_x(\mathcal{M}^{(n+2)}, g) = Q \cdot \mathrm{Hol}_{\widetilde{x}}(\widetilde{\mathcal{M}}^{(n+2)}, \widetilde{g}) = Q \cdot \mathrm{Hol}_x^0(\mathcal{M}^{(n+2)}, g),$$

where $\Phi : \widetilde{\mathcal{M}} \longrightarrow \mathcal{M}$ denotes the universal covering, $\widetilde{x} = (u, v, p)$, $\Phi(\widetilde{x}) = x$, and

$$Q := \langle Q(\sigma) \mid \sigma \in \pi_1(\mathcal{M}) \rangle \subset \mathbb{R}^* \times \mathrm{O}(n)$$

with $Q(\sigma) := (a_\sigma, d\mu_{\sigma^{-1}}^{-1} \circ \mathcal{P}_\sigma^h)$. Here, $\mu_\sigma := \nu_\sigma(u, v, \cdot)$ and $\mathcal{P}_\sigma^h$ is the parallel transport w.r.t. h along some curve in $\mathcal{N}$ from p to $\mu_{\sigma^{-1}}(p)$.

1.2 SCREEN BUNDLES

We have already seen in the last section that the n-dimensional vector bundle $\Sigma = \mathbb{L}^\perp/\mathbb{L}$ has a close relation to the Lorentzian manifold with special holonomy it is defined for. As we will see it plays an essential role in the study of Lorentzian manifolds with special holonomy and it is worth to give it a distinguished name (see the definition below).

Screen Bundles and Screen Distributions

Note that Σ fits into the exact sequence

$$0 \longrightarrow \mathbb{L} \hookrightarrow \mathbb{L}^\perp \twoheadrightarrow \Sigma \longrightarrow 0. \tag{1.15}$$

A non-canonical splitting $s : \Sigma \longrightarrow \mathbb{L}^\perp$ of this sequence[7] gives rise to an n-dimensional subbundle $\mathbb{S} := s(\Sigma)$ of the tangent bundle $T\mathcal{M}$.

Definition 1.13. *Let $(\mathcal{M}^{(n+2)}, g)$ be a Lorentzian manifold with special holonomy. The n-dimensional vector bundle $\Sigma = \mathbb{L}^\perp/\mathbb{L}$ is called* **screen bundle** *of $(\mathcal{M}, g)$ and any realization $\mathbb{S}$ of Σ as a splitting of (1.15) is called a* **screen distribution**. *The holonomy $\mathrm{Hol}(\Sigma, \nabla^\Sigma)$ is called the* **screen holonomy**.

We stress that $\mathbb{S}$ is far from being unique and the existence of particular realizations of the screen bundle is an interesting question, as we will see later in this section.

As Σ, any screen distribution $\mathbb{S}$ comes with a connection $\nabla^{\mathbb{S}}$ induced by the Levi-Civita connection of g via

$$\nabla_X^{\mathbb{S}} S := \mathrm{pr}_{\mathbb{S}} \circ \nabla_X^g S \tag{1.16}$$

6 In [BLL14, Theorem 3] the 1-forms A_u within the metric g do not appear. However, following the argumentation in [BLL14] it is not hard to see that even with the A_u, every isometry has to have the form (1.14).

7 Using a partition of unity of $\mathcal{M}$ such a splitting always exists.

for $X \in \Gamma(T\mathcal{M})$, $S \in \Gamma(\mathbb{S})$ and a metric $h := g|_{\mathbb{S}\times\mathbb{S}}$. Taking into account the bundle isomorphism $\mathbb{S} \cong \Sigma$ induced by $[\cdot]$, we see that $\mathrm{Hol}(\Sigma, \nabla^{\Sigma}) \cong \mathrm{Hol}(\mathbb{S}, \nabla^{\mathbb{S}})$.

There is another description of a screen distribution $\mathbb{S}$ which turns out to be very useful in many cases. Assume that the line bundle $\mathbb{L}$ admits a global nowhere vanishing section $V \in \Gamma(T\mathcal{M})$, which is equivalent to $\mathrm{pr}_{\mathbb{R}}(\mathrm{Hol}(\mathcal{M}, g)) \subset \mathbb{R}^+$ or time-orientability of $(\mathcal{M}, g)$, cf. [BLL14, Proposition 2]. In this case, the bundle of light-like lines transversal to $\mathbb{L}^{\perp}$ has a global nowhere-vanishing section $Z \in \Gamma(T\mathcal{M})$ with $g(V, Z) = 1$. For a time-orientable Lorentzian manifold with special holonomy, a light-like vector field Z with $g(V, Z) = 1$ is called *screen vector field* due to the following correspondence:

$$Z \in \{\text{Screen vector fields}\} \overset{\cong}{\longleftrightarrow} \mathbb{S} = V^{\perp_g} \cap Z^{\perp_g} \in \{\text{Screen distributions}\} \tag{1.17}$$

The screen vector fields give in general a better chance to distinguish the possible realizations of the screen bundle Σ.

Since in this thesis we only consider Lorentzian manifolds with special holonomy which admit a global nowhere vanishing light-like vector field $V \in \Gamma(T\mathcal{M})$ which spans $\mathbb{L}$, we make the following commitment.

Assume henceforth that all considered Lorentzian manifolds are time-oriented.

Note that if $V \in \Gamma(\mathbb{L})$ exists, then V is necessarily *recurrent*, i.e.

$$\nabla^g V = \beta \otimes V \tag{1.18}$$

for some $\beta \in \Omega^1(\mathcal{M})$, since $\mathbb{L}$ is parallel. We stress that for some purposes it will be necessary to assume that the 1-form β satisfies $\ker \beta = \mathbb{L}^{\perp}$.

Definition 1.14 ([Lär11]). *A time-orientable Lorentzian manifold with special holonomy and* $\ker \beta = \mathbb{L}^{\perp}$ *is called* **decent**.

Given a Lorentzian manifold $(\mathcal{M}, g)$ with special holonomy and a screen vector field $Z \in \Gamma(T\mathcal{M})$ and hence a screen distribution $\mathbb{S} = V^{\perp_g} \cap Z^{\perp_g}$ we can define an *associated Riemannian metric* g^R on $\mathcal{M}$ by

$$g^R(V, \cdot) := g(Z, \cdot),\ g^R(Z, \cdot) := g(V, \cdot), g^R(X, \cdot) := g(X, \cdot) \text{ for } X \in \Gamma(\mathbb{S}) \tag{1.19}$$

and extension by linearity. At a first glance, this seems not to give anything new but in the next section we will see the high importance of this metric. Informally, its relevance comes from the fact that it connects the study of the Lorentzian manifolds to Riemannian geometry. A first property is the following: let ∇^R denote the Levi-Civita connection to g^R. Then we see that:

Lemma 1.15. *For a decent Lorentzian manifold, any screen vector field $Z \in \Gamma(T\mathcal{M})$ is geodesic w.r.t. its associated Riemannian metric g^R in* (1.19).

Proof. Let $\mathbb{S} := V^{\perp_g} \cap Z^{\perp_g}$. Since $\nabla^g_X Z \in \Gamma(\mathbb{S})$ for all $X \in \Gamma(\mathbb{L}^{\perp})$, we see that $[X, Z] \in \Gamma(\mathbb{L}^{\perp})$ and thus $\nabla^R_Z Z = 0$ by the Koszul formula for g^R and the fact that $\mathbb{L}^{\perp} = \ker g^R(Z, \cdot)$. □

Horizontal and Involutive Screen Distributions

Observe that both, $\mathbb{L}$ and $\mathbb{L}^{\perp}$ are integrable distributions since they are parallel w.r.t. the Levi-Civita connection of g. Throughout this thesis we usually denote with $\mathcal{L}_x$ and $\mathcal{L}_x^{\perp}$ the maximal integral manifolds to $\mathbb{L}$ and $\mathbb{L}^{\perp}$ through $x \in \mathcal{M}$, respectively. By the Frobenius theorem, these are given by

$$\begin{aligned}
\mathcal{L}_x &= \{y \in \mathcal{M} \mid \text{ex. } \gamma : [0,1] \longrightarrow \mathcal{M}, \gamma(0) = x, \gamma(1) = y \text{ and } \dot{\gamma} \in \mathbb{L}\}, \\
\mathcal{L}_x^{\perp} &= \{y \in \mathcal{M} \mid \text{ex. } \gamma : [0,1] \longrightarrow \mathcal{M}, \gamma(0) = x, \gamma(1) = y \text{ and } \dot{\gamma} \in \mathbb{L}^{\perp}\}.
\end{aligned}$$

Of course, a realization $\mathbb{S}$ of the screen bundle $\Sigma = \mathbb{L}^{\perp}/\mathbb{L}$ is in general not involutive. We thus make the following definitions.

Definition 1.16. *A realization $\mathbb{S}$ of the screen bundle Σ is called* **involutive** *or* **integrable** *if so is the distribution $\mathbb{S} \subset T\mathcal{M}$ itself and* **horizontal** *if $[\Gamma(\mathbb{L}), \Gamma(\mathbb{S})] \subset \Gamma(\mathbb{S})$.*

For example, in the Walker coordinates (1.12), a horizontal realization $\mathbb{S}$ is spanned by $\partial_1 + A_1\partial_v, \ldots, \partial_n + A_n\partial_v$ and if the coefficients A_i constitute an u-dependent family of closed 1-forms on $\{u, v \equiv const\}$, then $\mathbb{S}$ is involutive. The corresponding screen vector field is given by $Z = \partial_u - H\partial_v - 2\sum_{i,j=1}^{n} g^{ij} A_i \partial_j$, where (g^{ij}) is the inverse of (g_{ij}).

The following lemma provides a first characterization of horizontal and involutive screen distributions in terms of its associated screen vector field.

Lemma 1.17. *A realization $\mathbb{S}$ is involutive and horizontal if and only if $dZ^{\flat}|_{\mathbb{L}^{\perp}\wedge\mathbb{L}^{\perp}} = 0$, which is equivalent to $V^{\flat} \wedge dZ^{\flat} = 0$.*

Assume that the screen bundle $\Sigma \longrightarrow \mathcal{M}$ is globally trivializable, i.e. it admits n linearly independent global sections $\sigma_1, \ldots, \sigma_n \in \Gamma(\Sigma)$. Then we have another characterization by the following result.

Lemma 1.18. *Let $(\mathcal{M}^{(n+2)}, g)$ be a Lorentzian manifold with special holonomy, $\mathbb{S}$ be a screen distribution and $Z \in \Gamma(T\mathcal{M})$ the corresponding screen vector field. Assume that the screen bundle is globally trivial. Then there exists a global orthonormal frame field $S_1, \ldots, S_n$ of $\mathbb{S}$. Furthermore, $\mathbb{S}$ is horizontal if and only if the global 1-forms $\alpha^i \in \Omega^1(\mathcal{M})$ defined by*

$$\alpha^i := g(\nabla^g S_i, Z), \tag{1.20}$$

satisfy

$$\alpha^i(V) = \beta(V), \tag{1.21}$$

for β as in (1.18) *and $\mathbb{S}$ is involutive if and only if*

$$\alpha^i(S_j) - \alpha^j(S_i) = 0. \tag{1.22}$$

Proof. Let $\mathbb{S}$ be a screen distribution. Then the bundle projection $[\cdot] : \mathbb{S} \longrightarrow\!\!\!\!\rightarrow \Sigma$ can be used to define a global frame field $S_1, \ldots, S_n$ for $\mathbb{S}$ from a trivialization of Σ. Then, since $[V, S_i] = \alpha^i(V) - \beta(S_i)$, the identity $\alpha^i(V) = g(\nabla^g_V S_i, Z) = g([V, S_i], Z)$ shows this first equivalence and $\alpha^i(S_j) - \alpha^j(S_i) = g([S_j, S_i], Z)$ the second one. $\square$

Now we compute the difference between two screen vector fields and their screen distributions, still under the assumption that the screen bundle Σ is globally trivial. Then, if $\mathbb{S}$ and $\widehat{\mathbb{S}}$ are two screen distributions, the sections σ_i define sections $S_i \in \Gamma(\mathbb{S})$ and $\widehat{S}_i \in \Gamma(\widehat{\mathbb{S}})$, $i = 1, \dots, n$, both orthonormal with respect to g, which are related by

$$\widehat{S}_i = S_i - b^i V \longmapsto \sigma_i = [S_i] \in \Gamma(\Sigma),$$

for smooth functions $b^i \in C^\infty(\mathcal{M})$. The corresponding screen vector fields Z and $\widehat{Z}$ are then related by

$$\widehat{Z} = Z + \sum_{k=1}^{n} b^k S_k - \frac{1}{2} \sum_{k=1}^{n} (b^k)^2 V,$$

and for the differentials of the duals we get

$$d\widehat{Z}^\flat = dZ^\flat + \sum_{k=1}^{n} \left(db^k \wedge S_k^\flat + b^k dS_k^\flat - b^k db^k \wedge V^\flat - \tfrac{1}{2}(b^k)^2 \beta \wedge V^\flat \right),$$

Then, computing the differentials of $Z^\flat$ and $dS_i^\flat$ we get

$$dZ^\flat = S_k^\flat \wedge \alpha^k + Z^\flat \wedge \beta$$

with α^k defined in (1.20), and

$$dS_i^\flat = \omega_i^{\ k} \wedge S_k^\flat + \alpha^i \wedge V^\flat,$$

where $\omega_i^{\ j}$ is the part of the connection 1-form defined by

$$\omega_i^{\ j} := g(\nabla^g S_i, S_j). \tag{1.23}$$

This allows us to express the differential of $\widehat{Z}^\flat$ in terms of a basis of the old screen, its connection coefficients and the functions b^i as

$$d\widehat{Z}^\flat = Z^\flat \wedge \beta + (db^k - \alpha^k + b^l \omega_l^{\ k}) \wedge S_k^\flat + b^k(\alpha^k - db^k - \tfrac{1}{2} b^k \beta) \wedge V^\flat,$$

in which we omit the sum symbol and use the summation convention. This, together with Lemma 1.17 gives us

Proposition 1.19. *Let $(\mathcal{M}^{(n+2)}, g)$ be a decent Lorentzian manifold. Assume that the screen bundle is globally trivial and a realization $\mathbb{S}$ is defined by global sections $S_1, \dots, S_n$ with associated screen vector field Z. Let α^i and $\omega_i^{\ j}$ be the corresponding connection forms defined in* (1.20) *and* (1.23)*. Then there is an involutive and horizontal screen distribution if and only if there are smooth functions $b^1, \dots, b^n$ on $\mathcal{M}$ which are solutions to the differential system*

$$0 = (db^k - \alpha^k + b^l \omega_l^{\ k}) \wedge S_k^\flat \text{ on } \mathbb{L}^\perp \wedge \mathbb{L}^\perp. \tag{1.24}$$

In particular, if there exist functions b^i such that

$$(db^i - \alpha^i + b^k \omega_k^{\ i}) = 0 \text{ on } \mathbb{L}^\perp, \tag{1.25}$$

then there is a horizontal and involutive screen distribution spanned by $S_i - b^i V$.

Finally, we arrive at a third characterization for horizontality involving the Riemannian metric g^R associated to a given screen distribution.

Proposition 1.20. *Let* $(\mathcal{M}^{(n+2)}, g)$ *be a decent Lorentzian manifold and* $\mathbb{S}$ *a screen distribution. Denote with* g^R *be the corresponding Riemannian metric defined in* (1.19)*. Then* $\mathbb{S}$ *is horizontal if and only if*

$$(\mathscr{L}_V g^R)|_{\mathbb{L}^\perp \times \mathbb{L}^\perp} = 0,$$

where $\mathscr{L}$ *denotes the Lie derivative.*

Proof. Since g^R and g coincide on $\mathbb{S} \times \mathbb{S}$ we have that $(\mathscr{L}_V g^R)(X, Y) = (\mathscr{L}_V g)(X, Y) = 0$ for all $X, Y \in \Gamma(\mathbb{S})$. Hence, the only non-vanishing terms of $(\mathscr{L}_V g^R)|_{\mathbb{L}^\perp \times \mathbb{L}^\perp}$ are

$$(\mathscr{L}_V g^R)(V, X) = g^R([X, V], V) = g([X, V], Z)$$

for $X \in \Gamma(\mathbb{S})$. But these vanish if and only if $\mathbb{S}$ is horizontal. □

For the remainder of this section we fix a maximal integral manifold $\mathcal{L}^\perp$ to the ∇^g-parallel distribution $\mathbb{L}^\perp$. By a screen distribution $\mathbb{S}$ *along* $\mathcal{L}^\perp$ we mean the vector bundle $\mathbb{S} \longrightarrow \mathcal{L}^\perp$ obtained by restricting $\mathbb{S}$ to $\mathcal{L}^\perp$. If such a screen along $\mathcal{L}^\perp$ is *involutive and horizontal* then this has deep consequences on the geometry of the associated Riemannian metric g^R, restricted to $\mathcal{L}^\perp$. Let h denote this metric on $\mathcal{L}^\perp$. To compare the induced connection $\nabla^g|_{\mathcal{L}^\perp}$ on $\mathcal{L}^\perp$ by g and ∇^h, we define the *potential* $\mathfrak{P}$ as

$$\mathfrak{P} := \nabla^g|_{\mathcal{L}^\perp} - \nabla^h. \tag{1.26}$$

Then we find:

Proposition 1.21. *Let* $(\mathcal{M}^{(n+2)}, g)$ *be a Lorentzian manifold with special holonomy and* V *a global nowhere vanishing light-like vector field spanning* $\mathbb{L}$*. Assume that, along* $\mathcal{L}^\perp$*, there exists a horizontal and involutive realization* $\mathbb{S}$ *of the screen bundle. Then the potential* $\mathfrak{P}$ *takes only values in* $\mathbb{L}$ *and the recurrent vector field* V *on* $\mathcal{M}$ *is parallel w.r.t. metric* h *on* $\mathcal{L}^\perp$*. In particular this implies that*

$$[R^g(X, Y) - R^h(X, Y)]W \in \Gamma(\mathbb{V}) \text{ for all } X, Y, W \in \Gamma(\mathbb{L}^\perp) \tag{1.27}$$

and for the Ricci curvature $\mathrm{Ric}^g|_{\mathbb{L}^\perp \times \mathbb{L}^\perp} = \mathrm{Ric}^h$.

Proof. For $V \in \Gamma(T\mathcal{M})$ we have that $\nabla^g V = \beta \otimes V$ for some $\beta \in \Omega^1(\mathcal{M})$. Since $h(V, V) = 1$, see (1.19), we obtain $h(\nabla^h V, V) = 0$. Moreover, applying the horizontality and involutivity property of $\mathbb{S}$, we see by the Koszul formula for h that

$$h(\nabla^h_{S_1} V, S_2) = g(\nabla^g_{S_1} V, S_2)$$

for all $S_1, S_2 \in \Gamma(\mathbb{S})$. This proves $\nabla^h V = 0$. Moreover, as $h(\nabla^h_{S_1} S_2, S_3) = g(\nabla^g_{S_1} S_2, S_3)$ for all $S_i \in \Gamma(\mathbb{S})$ we obtain $\mathfrak{P}(X, Y) \in \Gamma(\mathbb{L})$ for all $X, Y \in \Gamma(T\mathcal{L}^\perp)$. □

Imposing on the screen bundle Σ not only that it is trivial but further that its *screen holonomy is trivial*, we can strengthen the previous proposition to the following result.

Proposition 1.22. *Let $(\mathcal{M}^{(n+2)}, g)$ be a Lorentzian manifold with special holonomy and trivial screen holonomy. Assume that, along $\mathcal{L}^\perp$, there exists a horizontal and involutive realization $\mathbb{S}$ of the screen bundle. Then the Riemannian manifold $(\mathcal{L}^\perp, h)$ is flat and every ∇^Σ-parallel frame descends to a h-parallel frame.*

Proof. Since Σ is assumed to have trivial holonomy, i.e. $\mathrm{Hol}(\Sigma, \nabla^\Sigma) = \{\mathrm{id}\}$, by the holonomy principle we find global basis sections σ_i of Σ such that $\nabla^\Sigma \sigma_i = 0$, $i = 1, \ldots, n$. Hence, for any given screen distribution $\mathbb{S}$, the induced frame fields S_i satisfy $[\nabla^g_X S_i] = \nabla^\Sigma_X \sigma_i = 0$ and thus $\nabla^g S_i = \alpha^i \otimes V$. Therefore:

$$g(\nabla^g_X S_i, Y) = 0$$

for all $X, Y \in \Gamma(T\mathcal{L}^\perp)$. Writing out the Koszul formula for this term we get

$$\begin{aligned} 0 &= X(g(S_i, Y)) + S_i(g(X, Y)) - Y(g(S_i, X)) \\ &\quad + g([X, S_i], Y) + g([Y, S_i], X) + g([Y, X], S_i) \end{aligned}$$

This equation holds for all $X, Y \in \Gamma(T\mathcal{L}^\perp)$, but, since $\mathbb{S}$ was assumed to be horizontal and involutive, we have that the brackets $[X, S_i]$, $[Y, S_i]$ and $[X, Y]$ are in $\mathbb{S}$. Hence, when recalling the definition of h in (1.19), in the above expression we can replace the metric g by the Riemannian metric h on $\mathcal{L}^\perp$, which shows that

$$h(\nabla^h_X S_i, Y) = 0.$$

Hence, the S_i are parallel vector fields on $(\mathcal{L}^\perp, h)$. But we have already seen in Proposition 1.21 that V is also parallel for h. Hence, we have a h-orthonormal frame of $\mathcal{L}^\perp$ which is parallel for ∇^h yielding the flatness of $(\mathcal{L}^\perp, h)$. □

1.3 FOLIATIONS

We point out that the literature knows various different but equivalent definitions for foliations, see e.g. [MM03, Section 1.2]. We thus choose the definition which is best-suited for our purposes and follow [CC00].

General Facts

Let $\mathcal{M}$ be an $(n+2)$-dimensional manifold and let $q \in \mathbb{N}$ with $0 < q < n+2$. A *foliated chart of codimension* q is a pair $(\mathcal{U}, \varphi)$ consisting of an open subset $\mathcal{U} \subset \mathcal{M}$ and a diffeomorphism

$$\varphi : \mathcal{U} \subset \mathcal{M} \longrightarrow \mathcal{V}_\perp \times \mathcal{V}_\pitchfork \subset \mathbb{R}^{n+2} = \mathbb{R}^{n+2-q} \times \mathbb{R}^q \tag{1.28}$$

with $\mathcal{V}_\perp \subset \mathbb{R}^{n+2-q}$ and $\mathcal{V}_\pitchfork \subset \mathbb{R}^q$. A set

$$\mathcal{Q}_c := \varphi^{-1}(\mathcal{V}_\perp \times \{c\})$$

for $c \in \mathbb{R}^q$ is called a *plaque* of the foliated chart $(\mathcal{U}, \varphi)$.

Definition 1.23. *Let $\mathscr{F} = \bigsqcup_{\lambda\in\Lambda} \mathcal{L}_\lambda$ be a disjoint union of connected, immersed submanifolds of dimension $n+2-q$. Then $\mathscr{F}$ is said to be a* **foliation of $\mathcal{M}$ of codimension** *q, if and only if there exists an atlas $\mathfrak{A}$ of foliated charts such that for each $\lambda \in \Lambda$ and $(\mathcal{U},\varphi) \in \mathfrak{A}$, the intersection $\mathcal{L}_\lambda \cap \mathcal{U}$ (if non-empty) is a union of plaques.*

We see that, given a foliated chart $(\mathcal{U},\varphi)$, its plaques $\mathcal{Q}_\mathcal{U}$ are the connected components of the intersection of $\mathcal{U}$ with a leaf. For a foliated manifold $\mathcal{M}$ by $\mathscr{F}$ we write for short $(\mathcal{M},\mathscr{F})$ and with $T\mathscr{F}$ we denote the distribution induced by the tangent spaces of the leaves of $\mathscr{F}$.

Since we will make use of it, we point out that for foliations there another concept of holonomy called the *leaf holonomy*. To be self-contained and avoid confusion we will very briefly introduce this concept here, while there is no need to go into details. Namely, we only use it once in Chapter 3 where we use a classical result about foliations with trivial holonomy. An elaborated introduction can be found e.g. in [CC00, MM03].

The basic idea of the holonomy of a foliation $\mathscr{F}$ is to encode information about the behavior nearby curves $\gamma : [0,1] \longrightarrow \mathcal{L}$ inside of leaves. To achieve this, let $\gamma(0) = x_0$ and $\gamma(1) = x_1$. For simplicity we may assume that there is a foliated chart $(\mathcal{U},\varphi)$ such that $\gamma([0,1]) \subset \mathcal{U}$. Hence, $\operatorname{im}\gamma \subset \mathcal{Q}_\mathcal{U}$. Let $\mathcal{T}_0$ and $\mathcal{T}_1$ be transversals through x_0 resp. x_1, i.e. $\mathcal{T}_i = \varphi^{-1}(\{b_i\} \times \mathcal{V}_\pitchfork)$ for some $b_i \in \mathbb{R}^{n+2-q}$. Then there is a local diffeomorphism $h_\mathcal{U} : \mathcal{U}_0 \subset \mathcal{T}_0 \longrightarrow \mathcal{U}_1 \subset \mathcal{T}_1$ with $h(x_0) = x_1$ and $\mathrm{pr}_{\mathbb{R}^q} \circ h \circ \varphi|_{\mathcal{U}_0} = \mathrm{pr}_{\mathbb{R}^q} \circ \varphi|_{\mathcal{U}_0}$. Choosing another foliated chart $(\mathcal{V},\psi)$ the maps $h_\mathcal{U}$ and $h_\mathcal{V}$ coincide on some small neighborhood and thus, γ determines a germ h_γ at x_0. This germ now satisfies some relevant properties: it is independent of the chosen transversals, homotopic curves give rise to the same h_γ and if $\delta : [0,1] \longrightarrow \mathcal{L}$ is another path from x_1 to x_2 then $h_{\delta*\gamma} = h_\delta \circ h_\gamma$, where $\delta * \gamma$ is the concatenation of δ and γ. This together gives a homomorphism

$$\mathrm{hol} : \pi_1(\mathcal{L}, x_0) \longrightarrow \{\text{Germs of diffeomorphisms of } \mathbb{R}^q \text{ based at } 0\},$$

whose image $\mathrm{Hol}(\mathcal{L}) := \mathrm{hol}(\pi_1(\mathcal{L}, x_0))$ is the *leaf holonomy* of the foliation $\mathscr{F}$.

Lorentzian manifolds with special holonomy own a codimension one foliation induced by $\mathbb{L}^\perp$. If they are further decent, i.e. admit a global recurrent light-like vector field V with $\nabla^g V = \beta \otimes V$ and $\ker\beta = \mathbb{L}^\perp$ then this foliation is given by the kernel of a closed one form $V^\flat$. Therefore we may summarize some results dealing with this class of foliations.

Theorem 1.24. *Let $(\mathcal{M},\mathscr{F})$ be a foliated manifold, where $\mathscr{F}$ is defined by a closed non-singular 1-form. Then $\mathscr{F}$ exhibits the following properties.*

(i) All leaves of $\mathscr{F}$ have trivial lead holonomy [CN85, p. 80].

(ii) If $\mathcal{M}$ is closed then all leaves are diffeomorphic [Ton97, Corollary 3.31].

(iii) If all leaves are compact, the foliation arises as the fibers of a fibration $\mathcal{M} \longrightarrow \mathcal{M}/\mathscr{F}$ with $\mathcal{M}/\mathscr{F} \in \{\mathbb{R}, \mathbb{S}^1\}$ [Sha97, Corollary 8.6].[8]

Let $(\mathcal{M},\mathscr{F})$ be a foliated manifold and denote with $Q := T\mathcal{M}/T\mathscr{F}$ its *normal bundle*. A codimension one foliation $\mathscr{F}$ on $\mathcal{M}$ is called *transversally parallelizable*, if there is a

8 With $\mathcal{M}/\mathscr{F}$ we denote the *leaf space* which is the topological space $\mathcal{M}/_\sim$ with $x \sim y$ if and only if they are contained in a common leaf.

global vector field $Z \in \Gamma(T\mathcal{M})$ such that $[T\mathscr{F}, Z] \subset T\mathscr{F}$ and $Q = \mathbb{R} \cdot \mathrm{pr}_Q(Z)$ and it is called *complete* transversally parallelizable if Z is complete. We summarize the results of [Con74] about transversally parallelizable foliations in the following theorem.

Theorem 1.25. *Let $(\mathcal{M}, \mathscr{F})$ be a foliated manifold and $\mathscr{F}$ a complete transversally parallelizable codimension one foliation. Then it holds:*

(i) All leaves are diffeomorphic.

(ii) Either all leaves are closed or all leaves are dense in $\mathcal{M}$.

(iii) The universal cover of $\mathcal{M}$ is diffeomorphic to $\mathbb{R} \times \widetilde{\mathcal{L}}$, where $\widetilde{\mathcal{L}}$ is the universal cover of a distinguished leaf.

In the setting of decent Lorentzian manifolds, it is not hard so see that the foliation induced by $\ker V^\flat$ is transversally parallelizable, where *any* screen vector field provides a required transversal section $Z \in \Gamma(T\mathcal{M})$ [Lär11, Lemma 2.47].

Foliations and Riemannian Geometry

Foliations show their importance in particular in combination with certain Riemannian metrics on the foliated manifold $\mathcal{M}$. As we will see, the existence of Riemannian metrics with particular properties is in close coherence to the structure and geometry of the foliation. Let us begin with an important observation which takes a key role in many results of this thesis. Although its seems to be well-known [Con08, Example 10.4.28] we could not find a reference including a complete proof, hence we present it here.

Proposition 1.26. *Let $(\mathcal{M}, h)$ be a Riemannian manifold and $\mathscr{F}$ a foliation of $\mathcal{M}$. If $(\mathcal{M}, h)$ is geodesically complete, then so is every leaf $\mathcal{L}$ of the foliation w.r.t. its induced Riemannian metric.*

Proof. Let d and $d_{\mathcal{L}}$ denote the Riemannian distance functions w.r.t. h and $h_{\mathcal{L}}$, respectively, where $h_{\mathcal{L}}$ is the induced Riemannian metric on the immersed submanifold $\mathcal{L}$ of $\mathcal{M}$. Let $\{x_k\}_{k\in\mathbb{N}}$ be a $d_{\mathcal{L}}$-Cauchy sequence in $\mathcal{L}$. Since $d \leq d_{\mathcal{L}}$, the sequence $\{x_k\}_{k\in\mathbb{N}}$ is also d-Cauchy and hence it converges to some $x \in \mathcal{M}$.

Let $(\mathcal{U}, \varphi)$ and $(\mathcal{V}, \psi)$ denote foliated charts around x, s.t. $\mathcal{U} \subset \overline{\mathcal{U}} \subset \mathcal{V}$ and $\psi|_{\mathcal{U}} = \phi$, cf. [CC00, Lemma 11.2.9]. Hence, if $\mathcal{Q}_{\mathcal{U}}$, $\mathcal{Q}_{\mathcal{V}}$ are plaques of $\mathcal{U}$ resp. $\mathcal{V}$, it holds

$$\mathcal{Q}_{\mathcal{U}} \subset \overline{\mathcal{Q}_{\mathcal{U}}} \subset \mathcal{Q}_{\mathcal{V}}. \tag{1.29}$$

Of course, this implies that if we could show that all but finitely many points x_k lie in a single plaque $\mathcal{Q}_{\mathcal{U}}$, then some plaque $\mathcal{Q}_{\mathcal{V}}$ must also contain x and hence so does the leaf $\mathcal{L}$.

Let $\varepsilon > 0$ s.t. the geodesic ball $B_\varepsilon(x) = \{y \in \mathcal{M} \mid d(x,y) < \varepsilon\}$ of radius ε is contained in $\mathcal{U}$. Moreover, let $n_0 \in \mathbb{N}$ s.t. $d(x, x_k) < \frac{\varepsilon}{2}$ and $d_{\mathcal{L}}(x_k, x_l) < \frac{\varepsilon}{2}$ for all $k, l \geq n_0$. Suppose that x_k and x_l for $k \neq l$ and $k, l \geq n_0$ lie in different plaques. Then every path in $\mathcal{L}$ connecting x_k and x_l must leave $B_\varepsilon(x)$ since otherwise x_k and x_l would be contained in the same connected component of $B_\varepsilon(x) \cap \mathcal{L}$ and hence also in a common plaque $\mathcal{Q}_{\mathcal{U}}$, a contradiction. Since $d_{\mathcal{L}}(x_k, x_l) < \frac{\varepsilon}{2}$, there is a path γ in $\mathcal{L}$ connecting x_k and x_l whose length $\ell(\gamma)$ is bounded by $\frac{\varepsilon}{2}$. Let δ be the piecewise smooth path in $\mathcal{M}$ constructed by first connecting x and x_k via a minimizing h-geodesic σ and then connecting x_k and x_l via γ. One the one hand, δ is a path connecting x with x_l but leaving $B_\varepsilon(x)$ whence $\ell(\delta) \geq \varepsilon$. On the other hand,

$$\ell(\delta) = \ell(\sigma) + \ell(\gamma) = d(x, x_k) + \ell(\gamma) < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon$$

and we arrive in a contradiction. We deduce that all x_k, $k \geq n_0$, lie in a single plaque $\mathcal{Q}_{\mathcal{U}}$ and this completes the proof. □

The previous result involves a certain property of the foliation charts of a foliated manifold but makes at no other point use of an interplay of the Riemannian metric and the foliation. This is now changed in the following definition.

Definition 1.27. *Given a foliated manifold $(\mathcal{M}, \mathscr{F})$ and a Riemannian metric h on $\mathcal{M}$, then h is said to be* **bundle-like** *for $(\mathcal{M}, \mathscr{F})$, if and only if $(\mathscr{L}_X h)(Z_1, Z_2) = 0$ for all $X \in \Gamma(T\mathscr{F})$ and $Z_i \in \Gamma(T\mathscr{F}^{\perp_h})$. In this case, $(\mathcal{M}, \mathscr{F}, h)$ is called a* **Riemannian foliation**.

Given a foliated manifold $(\mathcal{M}, \mathscr{F}, h)$ with Riemannian metric h, the normal bundle $Q := T\mathcal{M}/T\mathscr{F}$ fits into an exact sequence

$$0 \longrightarrow T\mathscr{F} \longrightarrow T\mathcal{M} \xrightarrow{\pi} Q \longrightarrow 0$$

and for a splitting $s : Q \to T\mathscr{F}^{\perp_h}$ the metric h induces a *transversal metric* h^T on Q by $h^T := s^* h|_{T\mathscr{F}^{\perp_h}}$. We define a connection on Q by

$$\nabla^T_X \varphi := \begin{cases} \pi(\nabla^h_X Y_\varphi), & X \in \Gamma(T\mathscr{F}^{\perp_h}), \\ \pi([X, Y_\varphi]), & X \in \Gamma(T\mathscr{F}), \end{cases} \tag{1.30}$$

for any $\varphi \in \Gamma(Q)$ and $s(\varphi) = Y_\varphi$. It is torsion-free [Ton97, Proposition 3.8] and if h is bundle-like, it is metric [Ton97, Theorem 5.8]. Moreover, if h_1 and h_2 are two bundle-like metrics w.r.t. $(\mathcal{M}, \mathscr{F})$ such that $h_1^T = h_2^T$, then $\nabla_1^T = \nabla_2^T$ [Ton97, Theorem 5.9].

Definition 1.28. *Given a Riemannian foliation $(\mathcal{M}, \mathscr{F}, h)$ the connection $\nabla^T : \Gamma(Q) \longrightarrow \Gamma(T^*\mathcal{M} \otimes Q)$ is called the* **transversal Levi-Civita connection** *of the foliation.*

Given the transversal Levi-Civita connection ∇^T of a Riemannian foliation one obtains the corresponding curvature tensors R^T. Considering the bundle isomorphism $s(Q) \simeq Q$ one obtains the transversal Ricci curvature Ric^T defined through

$$\mathrm{Ric}^T(e_i, e_j) := \sum_{k=1}^{\dim Q} R^T(e_i, e_k, e_k, e_j) \tag{1.31}$$

for an h^T-orthonormal frame $\varphi_\ell \in \Gamma(Q)$ and $s(\varphi_\ell) = e_\ell$, $\ell = 1, \ldots, \dim Q$.

Particularly interesting are also the following Riemannian foliations.

Definition 1.29. *A one-dimensional Riemannian foliation $(\mathcal{N}, \mathscr{F}, h)$ is called a* **Riemannain flow**. *If this flow is generated by a non-singular vector field $V \in \Gamma(T\mathcal{M})$ with $h(V, V) = 1$ then the* **mean curvature 1-form**[9] *$\kappa \in \Omega^1(\mathcal{N})$ is defined by $\kappa := h(\nabla^h_V V, \cdot)$.*

9 Indeed, κ can be defined more general for any foliation [Ton97, (3.20)]. However, we will use this only in the setting of Riemannian flows where κ can be shown to be of the form used in the definition.

Remark 1.30. *Note that* $\kappa = \mathscr{L}_V \chi$ *for* $\chi = h(V, \cdot)$.

Once more we will build a bridge to Lorentzian manifolds with special holonomy. Namely, given a global light-like vector field $V \in \Gamma(\mathbb{L})$, this induces a Riemannian flow on $\mathcal{M}$ and, in particular, on each leaf $\mathcal{L}^\perp$ of the foliation induced by $\mathbb{L}^\perp$. Let there be given any realization $\mathbb{S}$ of the screen bundle and g^R its associated Riemannian metric from (1.19). As on page 23, we denote with h the associated Riemannian metric on $\mathcal{L}^\perp$.

Proposition 1.31. *For each leaf* $\mathcal{L}^\perp$ *of* $\mathbb{L}^\perp$ *and any realization* $\mathbb{S}$ *of the screen bundle,* $(\mathcal{L}^\perp, \mathscr{F}, h)$ *is a Riemannian flow, where* $\mathscr{F}$ *is given by the flow of* V *restricted to* $\mathcal{L}^\perp$.

Proof. By the definition of h we have that $(\mathscr{L}_V h)|_{\mathbb{S}\times\mathbb{S}} = (\mathscr{L}_V g)|_{\mathbb{S}\times\mathbb{S}}$ and since $\nabla^g V \in \Gamma(\mathbb{L})$ and hence $g(\nabla^g V, X) = 0$ for all $X \in \Gamma(\mathbb{S})$ the assertion follows. □

A Riemannian foliation is said to be *transversally orientable*, if the normal bundle Q admits an orientation. For the case of the Riemannian flow $(\mathcal{L}^\perp, \mathscr{F}, h)$ obtained from Proposition 1.31 we obtain the following.

Lemma 1.32. *Let* $(\mathcal{M}, g)$ *be an oriented Lorentzian manifold with special holonomy. Then, for each leaf* $\mathcal{L}^\perp$ *of* $\mathbb{L}^\perp$ *and any realization* $\mathbb{S}$ *of the screen bundle, the Riemannian flow* $(\mathcal{L}^\perp, \mathscr{F}, h)$ *is transversally orientable. Moreover,* $\mathcal{L}^\perp$ *is orientable.*

Proof. The argument can be found in [Lär11, p. 78]. Namely, one can prove that $\mathrm{Hol}(\nabla^T) \subset \mathrm{SO}(\dim \mathbb{S})$ and hence Σ is orientable. In particular, each leaf $\mathcal{L}^\perp$ is orientable since any screen vector field $Z \in \Gamma(T\mathcal{M})$ defines a unit normal vector field. □

Let $(\mathcal{N}, \mathscr{F})$ be a flow, i.e. a one-dimensional foliation. We say that $(\mathcal{N}, \mathscr{F})$ is *geodesible* if there exists a Riemannian metric such that the leaves are geodesic. It is said to be *isometric* if there exists a Riemannian metric h such that the leaves are the integral curves of a non-singular h-Killing field. For a Riemannian flow we have the following equivalences, cf. [Car81, Car84] and [Ton97, Proposition 6.7].

Theorem 1.33. *Let* $(\mathcal{N}, \mathscr{F})$ *be a Riemannian flow given by a non-singular vector field* $X \in \Gamma(T\mathcal{N})$. *Then the following are equivalent.*

(i) $\mathscr{F}$ *is isometric.*

(ii) $\mathscr{F}$ *is geodesible.*

(iii) There exists $\mathbb{E} \subset T\mathcal{N}$ *such that* $[X, E] \in \Gamma(\mathbb{E})$ *for all* $E \in \Gamma(\mathbb{E})$.

In particular, a Riemannian flow is isometric if and only if $\kappa = 0$. Note that for the Riemannian flow $(\mathcal{L}^\perp, \mathscr{F}, h)$ from above, the theorem simplifies as follows.

Remark 1.34. *There exists a horizontal screen distribution along* $\mathcal{L}^\perp$ *if and only if on* $(\mathcal{L}^\perp, \mathscr{F}, h)$ *the vector field* V *is* h*-geodesic and* h*-Killing.*

Proof. By the Koszul formula for h, there exists a horizontal screen distribution along $\mathcal{L}^\perp$ if and only if $\kappa := h(\nabla^h_V V, \cdot)$ vanishes. Together with Proposition 1.20 this completes the proof. □

Basic and Twisted Cohomology

For a foliated manifold $(\mathcal{M}, \mathscr{F})$ there is a cohomology theory whose chain complexes are somehow adapted to the foliation, called the *basic cohomology* of the foliation (for a precise definition see below). As in the case of the usual de Rham-cohomology it can be used as another tool to bridge a gap between topology and geometry by using, for example, Bochner's technique. This motivates us to give its definition in this geometric attached thesis and as we will see in Chapter 3 the interplay between these fields is fruitful.

Again we give a brief introduction to this theory while providing only the background necessary to understand the facts used within this thesis. A comprehensive introduction can be found, e.g. in [Ton97, Chapter 4, Chapter 7] which is in general a recommendable source for an introduction into foliations theory.

Definition 1.35. *A k-form $\alpha \in \Omega^k(\mathcal{M})$ is called* **basic** *if and only if $X \lrcorner \alpha = 0$ and $\mathscr{L}_X \alpha = 0$ for all $X \in \Gamma(T\mathscr{F})$. The set of basic k-forms is denoted by $\Omega^k_B(\mathscr{F})$.*

Since $\mathscr{L}_X d\alpha = d\mathscr{L}_X \alpha = 0$ and $X \lrcorner d\alpha = \mathscr{L}_X \alpha - d(X \lrcorner \alpha) = 0$ for arbitrary $X \in \Gamma(T\mathscr{F})$ and $\alpha \in \Omega^k_B(\mathscr{F})$, we obtain a subcomplex $(\Omega^*_B(\mathscr{F}), d_B)$ of the de Rham-complex with differentials $d_B := d|_{\Omega^*_B(\mathscr{F})}$ and hence a corresponding cohomology $H^*_B(\mathscr{F})$.

Definition 1.36. *The groups $H^*_B(\mathscr{F})$ are called the* **basic cohomology** *of the foliated manifold $(\mathcal{M}, \mathscr{F})$.*

If $\mathcal{M}$ is closed, there is also a formal L^2-adjoint δ_B [Ton97, Theorem 7.10] to d_B and hence a *transversal Laplacian*

$$\Delta_B := d_B \delta_B + \delta_B d_B \tag{1.32}$$

on $\Omega^*_B(\mathscr{F})$. Naturally, we then say that a basic form $\alpha \in \Omega^k_B(\mathscr{F})$ is *basic-harmonic* if and only if $\Delta_B \alpha = 0$.

In general, the spaces $H^*_B(\mathscr{F})$ are not finite dimensional, but for Riemannian foliations on closed manifolds they are. This is due to a Hodge-decomposition theorem for basic cohomology [KT87], see also [Ton97, Theorems 7.22 + 7.51].

Theorem 1.37. *Let $(\mathcal{M}, \mathscr{F}, h)$ be a transversally oriented Riemannian foliation on a closed oriented manifold $\mathcal{M}$ with $\kappa \in \Omega^1_B(\mathcal{M})$. Then*

$$\Omega^r_B \cong d_B \oplus \operatorname{im} \delta_B \oplus \mathscr{H}^r_B,$$

where $\mathscr{H}^r_B$ is the finite-dimensional *space of basic-harmonic r-forms. Moreover the spaces $H^r_B(\mathscr{F})$ are isomorphic to $\mathscr{H}^r_B$.*

Turning our attention to Riemannian flows again we have the following important result relating the mean curvature 1-form with the basic-harmonic forms:

Theorem 1.38 ([Dom98, Mas00]). *Let $(\mathcal{N}, \mathscr{F}, h)$ be a Riemannian flow on a compact manifold $\mathcal{N}$. Then there exists a bundle-like metric $\widehat{h}$ on $\mathcal{N}$ such that κ is basic-harmonic and $h^T = \widehat{h}^T$.*

Moreover, the basic cohomology gives us in addition to Theorem 1.33 another equivalent condition for a Riemannian flow to be isometric, cf. [MS85, Theorem A] and [Ton97, Corollary 7.57].

Theorem 1.39. *Let $(\mathcal{N}, \mathscr{F}, h)$ be a Riemannian flow on a closed oriented manifold $\mathcal{N}$ of dimension $(n+1)$. Then $H^n_B(\mathscr{F}) \in \{0, \mathbb{R}\}$ and $\mathscr{F}$ is isometric if and only if $H^n_B(\mathscr{F}) = \mathbb{R}$.*

For transversally oriented *isometric* Riemannian foliations of codimension q on oriented closed manifolds, the basic cohomology groups satisfy the classical Poincaré duality, namely in this case we have that $H^r_B(\mathscr{F}) \cong H^{q-r}_B(\mathscr{F})$, cf. [Ton97, Corollary 7.58]. However, for non-isometric foliations this is not necessarily true. This motivates to introduce a modified cohomology to obtain a Poincaré duality even for non-isometric Riemannian foliations. This leads to the *twisted cohomology* of the foliation and is defined as follows.

Let $(\mathcal{M}, \mathscr{F}, h)$ be a Riemannian foliation with basic mean curvature 1-form κ, i.e. $\kappa \in \Omega^1_B(\mathcal{M})$. Then, the chain complex $(\Omega^*_B(\mathscr{F}), d_\kappa := d_B - \kappa \wedge \cdot)$ is a subcomplex of the de Rham-complex and thus defines a cohomology, denoted with $H^*_\kappa(\mathscr{F})$.

Definition 1.40. *The groups $H^*_\kappa(\mathscr{F})$ are called the* **twisted cohomology** *of the foliated manifold $(\mathcal{M}, \mathscr{F})$.*

For this cohomology there is also a Hodge-decomposition [Ton97, (7.52)] and thus it behaves in a nice way as to obtain a type of Poincaré duality when comparing the basic cohomology and the twisted cohomology [Ton97, Theorem 7.54].

Theorem 1.41. *Let $(\mathcal{M}, \mathscr{F}, h)$ be a transversally oriented Riemannian foliation of codimension q on an oriented closed manifold. Then $H^r_B(\mathscr{F}) \cong H^{q-r}_\kappa(\mathscr{F})$.*

We complete this section with an instrument that relates the basic and twisted cohomology groups of Riemannian flows $\mathscr{F}$ and its encapsulating manifold $\mathcal{N}$. Let $(\mathcal{N}, \mathscr{F}, h)$ be a Riemannian flow and assume that h is chosen such that κ is basic-harmonic. (Note that this is *always* feasible due to Theorem 1.38.) Then the following result provides a Gysin sequence [Pri01, Theorem 3.2].

Theorem 1.42. *Let $(\mathcal{N}, \mathscr{F}, h)$ be a Riemannian flow on the closed manifold $\mathcal{N}$ with basic-harmonic mean curvature 1-form $\kappa \in \Omega^1_B(B)$. Then there is the long exact sequence*

$$\ldots \longrightarrow H^r_B(\mathscr{F}) \longrightarrow H^r_{\mathrm{dR}}(\mathcal{N}) \longrightarrow H^{r-1}_\kappa(\mathscr{F}) \xrightarrow{\wedge[\mathfrak{e}]} H^{r+1}_B(\mathscr{F}) \longrightarrow \ldots \tag{1.33}$$

where $[\mathfrak{e}] \in H^2_{-\kappa}(\mathscr{F})$ is the Euler class defined through the relation $d\chi = \mathfrak{e} + \chi \wedge \kappa$ and χ is the h-dual to a non-singular vector field defining $\mathscr{F}$.

If $(\mathcal{N}, \mathscr{F}, h)$ is an *isometric* Riemannian flow, then we have seen that $\kappa = 0$. In this case, the previous theorem turns into the following [Ton97, Theorem 6.13].

Theorem 1.43. *Let $(\mathcal{N}, \mathscr{F}, h)$ be an isometric Riemannian flow on the closed manifold $\mathcal{N}$ and K a unit h-Killing field defining $\mathscr{F}$. Then there is the long exact sequence*

$$\ldots \longrightarrow H^r_B(\mathscr{F}) \longrightarrow H^r_{\mathrm{dR}}(\mathcal{N}) \xrightarrow{\iota_*} H^{r-1}_B(\mathscr{F}) \xrightarrow{\delta} H^{r+1}_B(\mathscr{F}) \longrightarrow \ldots \tag{1.34}$$

where $\iota(\cdot) := (K \lrcorner \cdot)$, $\delta := [d\chi \wedge]$ and $\chi = h(K, \cdot)$.

1.4 STRUCTURE RESULTS

In this section we will derive some criteria under which an $\mathbb{R}$-factor can be split from the universal cover of decent Lorentzian manifolds. This will help to understand their holonomy and to examine their geometry.

We start with the fact that a manifold $\mathcal{M}$ admitting a closed 1-form and a complete transversal vector field is universally covered by a manifold $\mathbb{R} \times \mathcal{N}$.

Proposition 1.44. *Let $\mathcal{M}$ be a manifold admitting a closed, nowhere vanishing one-form η. Assume that there is a complete vector field Z such that $\eta(Z) = 1$. Then the leaves of the distribution $\ker(\eta)$ are all diffeomorphic to each other under the flow $\{\phi_t\}_{t\in\mathbb{R}}$ of Z, and the universal cover $\widetilde{\mathcal{M}}$ of $\mathcal{M}$ is diffeomorphic to $\mathbb{R} \times \mathcal{N}$ via $\mathbb{R} \times \mathcal{N} \ni (u, p) \longmapsto \phi_u(p) \in \widetilde{\mathcal{M}}$, where $\mathcal{N}$ is the universal cover of a leaf of $\ker(\eta)$.*

Proof. The idea of the proof can be found in [Mil63, Theorem 3.1]. Since η is closed, the distribution $\ker(\eta)$ is involutive. For each $t \in \mathbb{R}$, the flow ϕ_t of the complete vector field Z is a diffeomorphism of $\mathcal{M}$. Since η is closed and $\eta(Z) = 1$ we get for the Lie derivative that

$$(\mathscr{L}_Z\eta)(X) = d\eta(Z, X) + X(\eta(Z)) \equiv 0,$$

for all $X \in \Gamma(T\mathcal{M})$. This proves that ϕ_t maps the leaves of the distribution $\ker(\eta)$ diffeomorphically onto each other.

Let $\widetilde{\eta}$ and $\widetilde{Z}$ be the lifts of η and Z to the universal cover $\widetilde{\mathcal{M}}$. Then $\widetilde{Z}$ is still a complete vector field with $\widetilde{\eta}(\widetilde{Z}) = 1$ and $d\widetilde{\eta} = 0$. Hence, there is a real function $f \in C^\infty(\mathcal{M})$ such that $\widetilde{\eta} = df$. Let $\widetilde{\phi}_t$, $t \in \mathbb{R}$, denote the flow of $\widetilde{Z}$. Then, for each $p \in \widetilde{\mathcal{M}}$, the function $\tau_p : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $\tau_p(t) := f(\widetilde{\phi}_t(p)) \in \mathbb{R}$ satisfies

$$\tau_p'(t) = df_{\widetilde{\phi}_t(p)}(\widetilde{Z}) = \widetilde{\eta}_{\widetilde{\phi}_t(p)}(\widetilde{Z}) = 1.$$

Hence, $\tau(t) = t + f(p)$. This shows that $f : \widetilde{\mathcal{M}} \to \mathbb{R}$ is surjective and that two level sets $\widetilde{\mathcal{N}_a} = f^{-1}(a)$, $a \in \mathbb{R}$, are diffeomorphic under the flow: $\widetilde{\phi}_{b-a}(\widetilde{\mathcal{N}_a}) = \widetilde{\mathcal{N}_b}$. We derive a diffeomorphism $\Phi : \mathbb{R} \times \widetilde{\mathcal{N}_0} \longrightarrow \widetilde{\mathcal{M}}$ via

$$\Phi(u, p) := \widetilde{\phi}_u(p)$$

whose inverse is given by

$$\Phi^{-1}(p) = (f(p), \widetilde{\phi}_{-f(p)}(p)) \in \mathbb{R} \times \widetilde{\mathcal{N}_0}.$$

Being simply-connected, $\widetilde{\mathcal{N}} := \widetilde{\mathcal{N}_0}$ is the universal cover of the leaves of $\ker(\eta)$. □

We stress that the previous result already is implied by Theorem 1.25*(iii)*. Namely, since η is closed and $\eta(Z) = 1$, we see that $0 = d\eta(X, Z) = \eta([Z, X])$ and hence $[Z, X] \in \ker\eta$ for all $X \in T\mathcal{M}$. Consequently, the foliation provided by $\ker\eta$ is complete transversally parallelizable. However, we will need the explicit construction of the covering map by the flow of Z and hence provided the explicit proof here.

If on $\mathcal{M}$ there exists a *complete* Riemannian metric s.t. there exists a Killing field Z with $\eta(Z) = 1$, then this can be even strengthened in the following way, see also [PZ13, Proposition 10.1].

Proposition 1.45. *Let $(\mathcal{M}, h)$ be a complete Riemannian manifold admitting a closed, nowhere vanishing one-form η. Assume that there is a h-Killing vector field Z such that $\eta(Z) = 1$. Then the map*

$$\mathbb{R} \times \mathcal{N} \ni (u, p) \longmapsto \phi_u(p) \in \mathcal{M} \tag{1.35}$$

is a covering, where $\{\phi_t\}_{t\in\mathbb{R}}$ denotes the flow of Z and $\mathcal{N}$ a leaf of $\ker\eta$.

Proof. Since Z is a Killing field w.r.t. complete metric h, it is complete. Let ψ denote the map (1.35) and define by

$$h_0 := du^2 \oplus h_{\mathcal{N}}$$

a Riemannian metric on $\mathbb{R} \times \mathcal{N}$, where $h_{\mathcal{N}}$ denotes the induced metric on a distinguished leaf $\mathcal{N}$ of $\ker\eta$ and du the coordinate 1-form on $\mathbb{R}$. The metric h_0 is complete since so is $h_{\mathcal{N}}$ by Proposition 1.26. Moreover, since Z is h-Killing, its flow preserves the leaves to $\ker\eta$ and ψ is a local isometry, i.e. $\psi^* h = h_0$. Hence, ψ is a covering. □

Turning our attention to Lorentzian manifolds with special holonomy we can apply the previous propositions if we assume that there exist a complete vector field $V \in \Gamma(\mathbb{L})$ and a complete screen vector field Z.

Proposition 1.46. *Let $(\mathcal{M}^{(n+2)}, g)$ be a decent Lorentzian manifold and $\mathcal{L}^\perp$ a leaf of $\mathbb{L}^\perp$. Then the universal cover is diffeomorphic to $\mathbb{R} \times \widetilde{\mathcal{L}^\perp}$, where $\widetilde{\mathcal{L}^\perp}$ is the universal cover of $\mathcal{L}^\perp$. If $\mathcal{M}$ is compact, then it is even covered by $\mathbb{R} \times \mathcal{L}^\perp$.*

Proof. Let $Z \in \Gamma(T\mathcal{M})$ be a screen vector field and denote with $\eta := V^\flat$ the g-dual to the recurrent vector field $V \in \Gamma(\mathbb{L})$ spanning $\mathbb{L}$. Then η is closed and $\eta(Z) = 1$, hence Proposition 1.44 applies.

Let $\mathcal{M}$ be compact and denote with g^R the Riemannian metric associated to Z, cf. (1.19). By Lemma 1.15 the vector field Z is geodesic w.r.t. g^R and hence, by Theorem 1.33, there exists a Riemannian metric $\widehat{h}$ on $\mathcal{M}$ turning Z into a Killing vector field. Applying Proposition 1.45 yields the asserted covering. □

Under the assumption of the existence of a horizontal and involutive screen distribution we can strengthen the result to the following.

Proposition 1.47. *Let $(\mathcal{M}^{(n+2)}, g)$ be a decent Lorentzian manifold and $\mathcal{L}^\perp$ a leaf of $\mathbb{L}^\perp$. Assume that, along $\mathcal{L}^\perp$, there exists a horizontal and involutive screen distribution $\mathbb{S}$. Then the universal cover is diffeomorphic to $\mathbb{R}^2 \times \widetilde{\mathcal{S}}$, where $\widetilde{\mathcal{S}}$ is the universal cover of a leaf $\mathcal{S}$ of $\mathbb{S}$. If $\mathcal{M}$ is compact then it is even covered by $\mathbb{R}^2 \times \mathcal{S}$.*

Proof. Denote with g^R the Riemannian metric associated to $\mathbb{S}$ and with h its restriction to $\mathcal{L}^\perp$. Then $\omega := h(V, \cdot)$ is a closed 1-form on $\mathcal{L}^\perp$ since $\mathbb{S}$ is horizontal and involutive. Thus, Proposition 1.44 yields the first assertion.

Let $\mathcal{M}$ be compact. Then g^R is complete and thus, so is h by Proposition 1.26. Moreover, by the horizontality of $\mathbb{S}$ we deduce that V is a h-Killing field by Proposition 1.20. As a consequence, we can apply Proposition 1.45 to the complete Riemannian manifold $(\mathcal{L}^\perp, h)$. □

Remark 1.48. *Let $\{\phi_t\}_{t\in\mathbb{R}}$ denote the flow of Z and $\{\varphi_t\}_{t\in V}$ the flow of V. We point out that in the case of Proposition 1.46 the covering map is given by the flow of Z, i.e.*

$$\mathbb{R} \times \mathcal{L}^{\perp} \ni (u, p) \mapsto \phi_u(p) \in \mathcal{M}.$$

In case of Proposition 1.47 it is given by the concatenation of ϕ and φ, i.e.

$$\mathbb{R}^2 \times \mathcal{S} \ni (u, v, q) \mapsto \phi_u(\varphi_v(q)) \ni \mathcal{M}.$$

Remark 1.49. *Note that for the case of compact $\mathcal{M}$ in Proposition 1.47 we actually need that the horizontal and involutive screen $\mathbb{S}$ is the restriction of a screen on $\mathcal{M}$. In fact, all other results in this chapter also hold, if $\mathbb{S}$ is only defined as a vector bundle on a leaf $\mathcal{L}^{\perp}$ isomorphic to $\Sigma|_{\mathcal{L}^{\perp}}$, i.e. a screen* on *$\mathcal{L}^{\perp}$.*

If there is only a horizontal and involutive screen $\mathbb{S}$ on $\mathcal{L}^{\perp}$ then, in order for the second part of Proposition 1.47 to hold, one needs to further require that $\mathcal{L}^{\perp}$ is compact.

Similar structural results where studied in [Lär11, Chapter 2]. Moreover, there one can find examples for Lorentzian manifolds admitting an involutive realization of the screen bundle. For example, every stably causal time-orientable Lorentzian manifold with special holonomy admits an involutive realization of the screen bundle [Lär11, Proposition 2.52]. Of course, every upcoming chapter of this thesis provides more examples.

2 LORENTZIAN MANIFOLDS WITH ABELIAN HOLONOMY

2.1 PP-WAVES

Throughout this chapter we will consider Lorentzian manifolds $(\mathcal{M}^{(n+2)}, g)$ whose parallel line bundle $\mathbb{L}$ admits a *parallel* light-like vector field $V \in \Gamma(\mathbb{L})$, i.e. with $\nabla^g V = 0$. Such a Lorentzian manifold is said to have *Abelian holonomy* if and only if its reduced holonomy group is contained in the Abelian ideal $\mathbb{R}^n \subset \mathrm{SO}^0(1, n+1)$. An equivalent notion is that of pp-waves [EK62] (*pp-wave* for *plane fronted with parallel rays*). Namely, we define:

Definition 2.1. *A Lorentzian manifold $(\mathcal{M}^{(n+2)}, g)$ is called a* **pp-wave**[1] *if it admits a global parallel light-like vector field and if its curvature tensor R^g satisfies*

$$R^g(U, W) = 0 \tag{2.1}$$

for all $U, W \in \Gamma(\mathbb{L}^\perp)$.

According to this definition we have the following characterization of pp-waves which bridges the gap to the title of this chapter.

Proposition 2.2. *Let $(\mathcal{M}^{(n+2)}, g)$ be a Lorentzian manifold with parallel light-like $V \in \Gamma(\mathbb{L})$.*

(i) The following statements are equivalent:

a) $(\mathcal{M}, g)$ is a pp-wave.

b) $(\mathcal{M}, g)$ has Abelian holonomy.

c) For all $W \in \Gamma(\mathbb{L}^\perp)$ and $X, Y \in \Gamma(T\mathcal{M})$ it holds $R^g(X, Y)W \in \Gamma(\mathbb{L})$.

d) The screen bundle (Σ, ∇^Σ) is flat, i.e. the curvature of ∇^Σ vanishes.

e) There exist local sections $S_1, \ldots S_n$ of $\mathbb{L}^\perp$ with $g(S_i, S_j) = \delta_{ij}$ and local 1-forms α^i such that $\nabla^g S_i = \alpha^i \otimes V$. In this case, the 1-forms satisfy $d\alpha^i|_{\mathbb{L}^\perp \wedge \mathbb{L}^\perp} = 0$.

(ii) The holonomy of ∇^Σ is trivial if and only if the holonomy of $(\mathcal{M}, g)$ is contained in $\mathbb{R}^n$.

The proof is a straightforward computation carried out in [Lei02, BLL14]. The property for the differentials of the α^i's follows from the following computation: Let Z be a screen vector field, $X \in \Gamma(\mathbb{L}^\perp)$ and S_i frame fields as in (i.e). Then

$$d\alpha^i(S_j, X) = g(R^g(S_j, X)S_i, Z) = g(R^g(S_i, Z)S_j, X) = d\alpha^j(S_i, Z)g(V, X) = 0,$$

for all $i, j = 1, \ldots, n$, i.e. $d\alpha^i|_{\mathbb{L}^\perp \wedge \mathbb{L}^\perp} = 0$.

1 In the following we will consider compact manifolds of this type. We are aware that for compact manifolds the term *wave* might not be appropriate, but we use this term since it is established in the literature for manifolds with the given curvature properties. Later we will see that an appropriate name would be *screen flat*, but this term has other obvious problems.

Clearly, manifolds with trivial screen holonomy are pp-waves, but for non simply-connected manifolds the converse is not true (see [BLL14] for examples).

Locally, for a pp-wave the coordinates in (1.12) can be chosen in a way such that $A_i \equiv 0$ and $h_{ij} \equiv \delta_{ij}$, i.e. with h being the standard flat metric for all u. Therefore, locally, g takes the form

$$g^H = 2dudv + 2Hdu^2 + \sum_{i,j=1}^{n} \delta_{ij} dx^i dx^j, \tag{2.2}$$

where $H = H(u, x^1, \dots, x^n)$ is a smooth function. In these coordinates, $\nabla^g \partial_v = 0$ and

$$\nabla^g_{\partial_i} \partial_j = 0,\ \nabla^g_{\partial_i} \partial_u = \partial_i(H)\partial_v,\ \nabla^g_{\partial_u} \partial_u = \partial_u(H)\partial_v - \sum_{i=1}^{n} \partial_i(H)\partial_i$$

which implies that the only non-vanishing curvature terms of g, up to symmetries, are

$$R^g(\partial_i, \partial_u, \partial_u, \partial_j) = -\partial_i \partial_j H, \tag{2.3}$$

and for the Ricci curvature,

$$Ric(\partial_u, \partial_u) = \Delta(H), \tag{2.4}$$

where $\Delta = -\sum_{i=1}^{n} \partial_i^2$ is the flat Laplacian. We refer to the Euclidean space $\mathbb{R}^{n+2}$ equipped with the metric g^H in (2.2) as *standard pp-wave*. Formula (2.3) shows that the connected holonomy of a pp-waves is *equal* to $\mathbb{R}^n$, and hence indecomposable, if there is a point in $\mathcal{M}$ with local coordinates such that the Hessian of H is non-degenerate at this point.

Since the distribution $\mathbb{L}^\perp$ is parallel and thus defines a foliation of $\mathcal{M}$ into totally geodesic leaves of codimension one, the flatness of the screen bundle can be stated as

Lemma 2.3. *A Lorentzian manifold $(\mathcal{M}^{(n+2)}, g)$ with parallel light-like vector field V is a pp-wave if and only if, for each leaf $\mathcal{L}^\perp$ of $\mathbb{L}^\perp$, the linear connection which is induced on $\mathcal{L}^\perp$ by the Levi-Civita connection of g is flat.*

Proof. Let $\nabla^{\mathcal{L}^\perp}$ be the linear connection defined by ∇^g on a leaf $\mathcal{L}^\perp$ of $\mathbb{L}^\perp$, i.e. $\nabla^{\mathcal{L}^\perp}_U W := \nabla^g_U W \in \Gamma(\mathbb{L}^\perp|_{\mathcal{L}^\perp})$ for $U, W \in \Gamma(T\mathcal{L}^\perp)$, where $T\mathcal{L}^\perp = \mathbb{L}^\perp|_{\mathcal{L}^\perp}$. Hence, for the curvature R^g of ∇^g and $R^{\mathcal{L}^\perp}$ of $\nabla^{\mathcal{L}^\perp}$ we have

$$R^{\mathcal{L}^\perp}(U, W)S = R^g(U, W)S$$

for all $U, W, S \in \Gamma(\mathbb{L}^\perp|_{\mathcal{L}^\perp})$. This term vanishes if and only if $g(R^g(U, W)S, X) = 0$ for all $X \in \Gamma(T\mathcal{M})$, which is equivalent to (2.1) in the Definition 2.1 of pp-waves. □

Recall that due to Proposition 1.22, Lorentzian manifolds with trivial screen holonomy which admit a horizontal and involutive realization $\mathbb{S}$ of the screen bundle have flat leaves w.r.t. the induced Riemannian metric induced by $\mathbb{S}$. Of course, the existence of such $\mathbb{S}$ can by formulated with the aid of Proposition 1.19 as follows.

Proposition 2.4. *Let $(\mathcal{M}^{(n+2)}, g)$ be a Lorentzian manifold with parallel light-like vector field V and trivial screen holonomy. Then, for each screen distribution $\mathbb{S} = V^{\perp_g} \cap Z^{\perp_g}$, there is a global frame field $S_1, \dots, S_n$ of $\mathbb{S}$ with $\nabla^g S_i = \alpha^i \otimes V$ and the α^i satisfy*

$$d\alpha^i(X, Y) = R^g(X, Y, S_i, Z),$$

for all $X, Y \in \Gamma(T\mathcal{M})$ and hence

$$d\alpha^i|_{\mathbb{L}^\perp \wedge \mathbb{L}^\perp} = 0.$$

Furthermore, given a screen distribution $\mathbb{S}$ and functions $b^1, \dots, b^n$ on $\mathcal{M}$ such that

$$(db^i - \alpha^i)|_{\mathbb{L}^\perp} = 0,$$

there is an involutive and horizontal screen distribution spanned by $S_i - b^i V$.

Proof. Since Σ is assumed to have trivial holonomy, the holonomy principle guarantees us global basis sections $\sigma_i \in \Gamma(\Sigma)$ such that $\nabla^\Sigma \sigma_i = 0$. Hence, for a given screen distribution, the induced frame fields S_i satisfy $[\nabla^g_X S_i] = \nabla^\Sigma_X \sigma_i = 0$ and thus

$$\nabla^g S_i = g(\nabla^g S_i, Z)V = \alpha^i \otimes V,$$

or equivalently, $\omega_i^j = 0$ (see (1.23) on page 22). This shows

$$R^g(X, Y)S_i = d\alpha_i(X, Y) \cdot V,$$

since V is parallel. As $(\mathcal{M}, g)$ has trivial screen holonomy and is hence a pp-wave, we obtain $d\alpha^i|_{\mathbb{L}^\perp \wedge \mathbb{L}^\perp} = 0$.

Given functions b^i with $(db^i - \alpha^i)|_{\mathbb{L}^\perp} = 0$, from $\omega_i^j = 0$ and equation (1.25) in Proposition 1.19 we see that $\widehat{S_k} = S_k - b^k V$ defines a horizontal and involutive screen distribution. □

2.2 UNIVERSAL COVER

The aim of this section is to study the universal cover of compact $(n+2)$-dimensional pp-waves $(\mathcal{M}^{(n+2)}, g)$. As we will see, compact pp-waves are aspherical[2] manifolds since their universal cover is diffeomorphic to $\mathbb{R}^{n+2}$. Further we show that the pull-back of g to the universal cover is globally isometric to a standard pp-wave metric (2.2).

We start with the observation that on the universal cover of a pp-wave there always exists a horizontal and involutive realization of the screen bundle.

Theorem 2.5. *Let $(\mathcal{M}^{(n+2)}, g)$ be a pp-wave with parallel light-like vector field $V \in \Gamma(T\mathcal{M})$ and complete screen vector field Z. Then there exists a horizontal and involutive screen distribution $\mathbb{S}$ on the universal cover $(\widetilde{\mathcal{M}}, \widetilde{g})$ of $(\mathcal{M}, g)$.*

In particular, there are n linear independent sections of $\mathbb{S}$ which are $\nabla^{\widetilde{g}}$-parallel along the leaves of $\widetilde{\mathbb{L}^\perp}$, where $\nabla^{\widetilde{g}}$ is the Levi-Civita connection of $\widetilde{g}$.

Proof. By a tilde we denote the lift of any object to the universal cover $\widetilde{\mathcal{M}}$ of $\mathcal{M}$. Let $\mathbb{S} = V^{\perp_g} \cap Z^{\perp_g}$ be a screen distribution defined by a complete screen vector field Z. By assumption, the bundle $\widetilde{\Sigma} \longrightarrow \widetilde{\mathcal{M}}$ is flat, and, since $\widetilde{\mathcal{M}}$ is simply-connected, has trivial holonomy. By the holonomy principle we derive n linearly independent global parallel

2 A manifold is said to be *aspherical* if all but the first homotopy group vanish. In other words, they are $K(G,1)$-spaces, i.e. the *Eilenberg-MacLane spaces* with fundamental group equal to some G and all other homotopy groups vanishing.

sections $\sigma_1, \ldots, \sigma_n \in \Gamma(\widetilde{\Sigma})$. These give rise to sections $S_1, \ldots, S_n \in \Gamma(\widetilde{\mathbb{S}})$ with $\nabla^{\widetilde{g}} S_i = \alpha^i \otimes \widetilde{V}$ where $\alpha^i := \widetilde{g}(\nabla^{\widetilde{g}} S_i, \widetilde{Z})$. Since $\mathcal{M}$ is a pp-wave, according to Proposition 2.4, they satisfy

$$d\alpha^i|_{\widetilde{\mathbb{L}^\perp} \wedge \widetilde{\mathbb{L}^\perp}} = 0. \tag{2.5}$$

By Proposition 1.46, the universal cover $\widetilde{\mathcal{M}}$ is diffeomorphic to $\mathbb{R} \times \mathcal{N}$, where $\mathcal{N} := \widetilde{\mathcal{L}^\perp}$ is the universal cover of the leaves of the distribution $\mathbb{L}^\perp$, and the map $\mathbb{R} \times \mathcal{N} \to \widetilde{\mathcal{M}}$ is given by the flow of $\widetilde{Z}$. Now, for each $r \in \mathbb{R}$, let

$$\begin{array}{rcl} \iota_{(r)} : \mathcal{N} & \hookrightarrow & \mathbb{R} \times \mathcal{N} \\ x & \mapsto & (r, x) \end{array}$$

denote the inclusion of $\mathcal{N}$ into $\mathbb{R} \times \mathcal{N}$. We use these to pull back the α^i's to $\mathcal{N}$,

$$\alpha^i_{(r)} := (\iota_{(r)})^* \alpha^i,$$

which is now a one-parameter family of one-forms on $\mathcal{N}$, depending smoothly on the parameter $r \in \mathbb{R}$. Because of equation (2.5), all $\alpha^i_{(r)}$ are closed,

$$d\alpha^i_{(r)} = d(\iota^*_{(r)} \alpha^i) = \iota^*_{(r)} d\alpha^i = 0.$$

Fixing $x_0 \in \mathcal{N}$, since $\mathcal{N}$ is simply-connected, for each $i = 1, \ldots, n$ and each $r \in \mathbb{R}$ we find a unique function $b^i_{(r)} \in C^\infty(\mathcal{N})$ such that

$$db^i_{(r)} = \alpha^i_{(r)}, \text{ and } b^i_{(r)}(x_0) = 0,$$

where the differential is the differential on $\mathcal{N}$. Hence we obtain smooth $b^i \in C^\infty(\mathbb{R} \times \mathcal{N})$ defined by

$$b^i(r, x) = b^i_{(r)}(x).$$

We have to verify that these functions are smooth on $\mathbb{R} \times \mathcal{N}$: Take an arbitrary $\widehat{x} \in \mathcal{N}$ and fix coordinates $(\mathcal{U}, \varphi = (x^1, \ldots, x^{n+1}))$ around $\widehat{x}$ such that $\varphi(\mathcal{U})$ is star-shaped and $\varphi(\widehat{x}) = 0$. Over $\mathcal{U}$ we write $\alpha^i_{(r)}$ as

$$\alpha^i_{(r)}|_{(r, \varphi^{-1}(x^1, \ldots, x^n))} = \sum_{k=1}^n \alpha^i_k(r, x^1, \ldots, x^n) dx^k$$

with α^i_k smooth functions on $\mathbb{R} \times \varphi(\mathcal{U})$ and the solutions $b^i_{(r)}$ are given by

$$\begin{array}{rcl} b^i(r, \varphi^{-1}(x^1, \ldots, x^n)) & = & b^i_{(r)} \circ \varphi^{-1}(x^1, \ldots, x^n) \\ & = & \displaystyle\sum_{k=1}^n x^k \int_0^1 \alpha^i_k(r, t(x^1, \ldots, x^n)) \mathrm{d}t + b^i(r, \widehat{x}), \end{array}$$

for all $(x^1, \ldots, x^n) \in \varphi(\mathcal{U})$. Since $\alpha^i_{(r)}$ and hence α^i_k depend smoothly on r, this is smooth in r and x^i, as soon as $b^i(\cdot, \hat{x})$ is smooth in r. Now choosing $\hat{x} = x_0$ first, we see that b^i is smooth on $\mathbb{R} \times \mathcal{U}$, where $\mathcal{U}$ is a star-shaped neighbourhood of $\hat{x} = x_0$. Then covering $\mathcal{N}$ by star-shaped neighbourhoods, this argument shows that b^i is smooth on $\mathbb{R} \times \mathcal{N}$.

Using these $b^i \in C^\infty(\mathbb{R} \times \mathcal{N})$ we define the new screen distribution as

$$\widehat{\mathbb{S}} := \operatorname{span}\{\widehat{S}_1, \dots, \widehat{S}_n\} \quad \text{with} \quad \widehat{S}_i := S_i - b^i \widetilde{V}.$$

For every $\widehat{X} = d\iota_{(r)}|_x(X) \in \mathbb{L}^\perp{}_{(r,x)} \subset T_{(r,x)}(\mathbb{R} \times \mathcal{N})$ with $X \in T_x\mathcal{N}$ the $\widehat{S}_i$ satisfy

$$\nabla^{\widetilde{g}}{}_{\widehat{X}}\widehat{S}_i \;=\; \left(\alpha^i(\widehat{X}) - db^i(\widehat{X})\right)\widetilde{V} \;=\; \left(\alpha^i_{(r)}(X) - db^i_{(r)}(X)\right)\widetilde{V} \;=\; 0,$$

which shows that $\widehat{\mathbb{S}}$ is involutive and horizontal. □

Let ∇^h denote the connection of the Riemannian metric (1.19) on the universal cover $\widetilde{\mathcal{M}}$ associated to $\mathbb{S}$, where $\mathbb{S}$ is the horizontal and involutive realization of the screen bundle obtained be the theorem. Due to Proposition 1.22, the sections obtained in the theorem which are $\nabla^{\widetilde{g}}$-parallel along the leaves of $\widetilde{\mathbb{L}^\perp}$, are also parallel w.r.t. ∇^h. Hence we obtain:

Corollary 2.6. *On the universal cover there exists a realization $\mathbb{S}$ of the screen bundle s.t. the flat connections $\nabla^{\widetilde{g}}|_{\widetilde{\mathcal{L}^\perp}}$ and ∇^h on $\widetilde{\mathcal{L}^\perp}$ coincide.*

The proof of the theorem also implies:

Corollary 2.7. *For any screen $\mathbb{S}$ on the universal cover of a pp-wave defined by a complete screen vector field and linearly independent sections $S_1, \dots, S_n$ of $\mathbb{S}$ with $\nabla^g S_i = \alpha^i \otimes \widetilde{V}$ there are smooth functions b^i such that $\widehat{\mathbb{S}}$ spanned by $\widehat{S}_i = S_i - b^i\widetilde{V}$ is involutive and horizontal.*

Remark 2.8. *The horizontal and involutive screen distribution on the universal cover obtained by this result does not necessarily descend to a horizontal and involutive one on the base manifold. In fact, Example 4.8 in Chapter 4 provides compact pp-waves for which no horizontal and involutive realization of the screen bundle exists.*

Combining Proposition 1.46 and the previous theorem gives us a description of the universal cover of certain pp-waves. The second part of the proof will follow ideas in [DR09].

Theorem 2.9. *Let $(\mathcal{M}^{(n+2)}, g)$ be a pp-wave with parallel light-like vector field $V \in \Gamma(T\mathcal{M})$ satisfying the following completeness assumptions:*

(i) The maximal geodesics along the leaves of $\mathbb{L}^\perp$ are defined on $\mathbb{R}$, and

(ii) there exists a complete screen vector field Z.

Then the universal cover $\widetilde{\mathcal{M}}$ of $\mathcal{M}$ is diffeomorphic to $\mathbb{R}^{n+2}$. Moreover, the universal cover $(\widetilde{\mathcal{M}}, \widetilde{g})$ is globally isometric to a standard pp-wave

$$\left(\mathbb{R}^{n+2},\; g^H = 2dudv + 2H(u, x^1, \dots, x^n)du^2 + \delta_{ij}dx^i dx^j\right). \tag{2.6}$$

Under this isometry, the lift of the parallel vector field V is mapped to ∂_v.

Proof. We first prove that the universal cover $\widetilde{\mathcal{M}}$ of $\mathcal{M}$ is diffeomorphic to $\mathbb{R}^{n+2}$ and then proceed to show that the lift of g is isometric to (2.6). By a tilde we shall denote the lift of an object to the universal cover. With Z we denote a complete screen vector field and with $\mathbb{S}$ the corresponding screen distribution on $\mathcal{M}$.

Step 1. Since Z is complete, we can apply Proposition 1.46 and obtain that the universal cover $\widetilde{\mathcal{M}}$ of $\mathcal{M}$ is diffeomorphic to $\mathbb{R} \times \widetilde{\mathcal{L}^\perp}$, where $\widetilde{\mathcal{L}^\perp}$ is the universal cover of a leaf $\mathcal{L}^\perp$ of the distribution $\mathbb{L}^\perp$ on $\mathcal{M}$. Clearly, $\widetilde{\mathcal{L}^\perp}$ is also a leaf of the distribution $\widetilde{\mathbb{L}^\perp}$ on $\widetilde{\mathcal{M}}$. Using assumption (i) and because $(\mathcal{M}, g)$ is a pp-wave we conclude that $\widetilde{\nabla}|_{\widetilde{\mathcal{N}}}$ is a complete and flat connection on the simply-connected manifold $\widetilde{\mathcal{N}}$. Hence the exponential map w.r.t. $\widetilde{\nabla}|_{\widetilde{\mathcal{N}}}$ is a diffeomorphism and thus $\widetilde{\mathcal{N}}$ diffeomorphic to $\mathbb{R}^{n+1}$. This proves the first part of the statement.

Step 2. The proof of the second part, that the lifted metric $\widetilde{g}$ is isometric to a standard pp-wave, is more involved and requires some auxiliary statements.

Lemma 2.10. *Let $(\mathcal{M}^n, \nabla)$ be a smooth n-dimensional manifold with a torsion free connection ∇ and let $\delta : I \longrightarrow \mathcal{M}$ be a curve in $\mathcal{M}$ with $0 \in I \subset \mathbb{R}$. Then a vector field $X \in \Gamma(\delta^* T\mathcal{M})$ along the curve δ is parallel along δ if and only if the vector field $Y \in \Gamma(\delta^* T\mathcal{M})$ with $Y(t) := t \cdot X(t)$ satisfies $\frac{\nabla^2}{dt^2} Y(t) \equiv 0$.*

Proof. One direction of the proof is trivial, so let us assume that

$$\frac{\nabla^2}{dt^2} Y(t) \equiv 0. \tag{2.7}$$

By the Leibniz rule this implies that

$$2 \cdot \frac{\nabla}{dt} X(t) + t \cdot \frac{\nabla^2}{dt^2} X(t) = 0 \tag{2.8}$$

for $t \in I$. Now let $E_i(t) := \mathcal{P}^{\nabla}_{\delta(t)}(e_i)$ with fixed $x^0 = \delta(0) \in \mathcal{M}$ and a basis $e_1, \ldots, e_n$ in $T_{x^0}\mathcal{M}$. Consequently, we can write

$$X(t) = \sum_{i=1}^{n} \xi_i(t) \cdot E_i(t), \tag{2.9}$$

which implies that X is parallel along δ if $\xi_i' \equiv 0$ on I for all $i = 1, \ldots, n$. If we write X in the form (2.9), formula (2.8) implies that the coefficient functions $\xi_i \in C^\infty(I)$ of X must satisfy the ordinary differential equation $2\xi_i'(t) + t \cdot \xi_i''(t) = 0$ with the initial values given by $X(0) \in T_{x^0}\mathcal{M}$. Each such equation only has the constant solution defined on I: A discussion of the solutions y on $(0, \infty) \cap I$ or $(-\infty, 0) \cap I$ of $2y(t) + ty'(t) = 0$ yields $|y(t)| = \frac{1}{t^2}$. Therefore, $y \equiv 0$ is the only solution, defined on 0 since otherwise it must be equal to $\pm\frac{1}{t^2}$ on $(0, \infty) \cap I$ or $(-\infty, 0) \cap I$ – a contradiction. Hence, if $\xi_i \in C^\infty(I)$ with $0 \in I$ solves $2\xi_i'(t) + t \cdot \xi_i''(t) = 0$, then $y := \xi_i'$ is defined on $0 \in I$ and solves the latter differential equation of order one. Consequently, $y = \xi_i'$ is identically zero. □

We are now going to construct the required isometry $\Phi : \mathbb{R}^{n+2} \longrightarrow \widetilde{\mathcal{M}}$. For that purpose let $\gamma : \mathbb{R} \longrightarrow \widetilde{\mathcal{M}}$ be the integral curve of the complete vector field $\widetilde{Z}$ through $x^0 \in \widetilde{\mathcal{M}}$ and $S_1, \ldots, S_n \in \Gamma(\widetilde{\mathbb{S}})$ such that $\widetilde{g}(S_i, S_j) = \delta_{ij}$ and $\nabla^{\widetilde{g}} S_i = \alpha_i \otimes \widetilde{V}$, see Proposition 2.4. We define the smooth map $\Phi : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \longrightarrow \widetilde{\mathcal{M}}$ by

$$\Phi(u, v, x^1, \ldots, x^n) := \exp_{\gamma(u)}(v \cdot \widetilde{V}(\gamma(u)) + \langle x, \vec{S}(u) \rangle), \tag{2.10}$$

where exp is the exponential map of $\widetilde{g}$ and where we define

$$\langle x, \vec{S}(u) \rangle := \sum_{k=1}^{n} x^k S_k(\gamma(u)).$$

Then we show

Lemma 2.11. *The smooth map Φ in (2.10) is a well-defined diffeomorphism.*

Proof. By the completeness assumption *(i)*, the exponential

$$\exp_p : \widetilde{\mathbb{L}}^{\perp}_p \cong T_p\mathcal{N} \longrightarrow \mathcal{N}$$

is defined on the whole tangent space for each leaf $\mathcal{N} := \widetilde{\mathcal{L}^{\perp}}$ through $p \in \widetilde{\mathcal{M}}$. Moreover, it is a diffeomorphism, since $\nabla^{\widetilde{g}}|_{\mathcal{N}}$ is a complete and flat connection on the simply-connected manifold $\mathcal{N}$. Hence, in order to prove that Φ is injective, it suffices to show that $\Phi(u_1, v_1, x) \neq \Phi(u_2, v_2, y)$ for all $v_1, v_2 \in \mathbb{R}$ and $x, y \in \mathbb{R}^n$ whenever $u_1 \neq u_2$. But for $u_1 \neq u_2$ we have

$$\gamma(u_1) \text{ and } \gamma(u_2) \text{ are contained in two disjoint leaves } \mathcal{N}_1 \text{ and } \mathcal{N}_2,$$

to $\widetilde{\mathbb{L}^{\perp}}$, respectively. To see this, recall that – as we have seen in the proof of Proposition 1.44 – it holds $\widetilde{\eta} = df$ for some smooth function f on $\widetilde{\mathcal{M}}$ and $\eta := V^{\flat}$ such that $f(\gamma(u)) = u + f(x^0)$. In this situation, each leaf is given as a level set of f and $\mathcal{N}_1 = f^{-1}(u_1 + f(x^0))$, $\mathcal{N}_2 = f^{-1}(u_2 + f(x^0))$ which implies that $\gamma(u_1)$ and $\gamma(u_2)$ cannot lie within the same leaf. But then we see that

$$\operatorname{im} \exp_{\gamma(u_1)} = \mathcal{N}_1 \quad \text{and} \quad \operatorname{im} \exp_{\gamma(u_2)} = \mathcal{N}_2$$

and since $\mathcal{N}_1 \cap \mathcal{N}_2 = \emptyset$, this yields $\Phi(u_1, v_1, x) \neq \Phi(u_2, v_2, y)$ for all $v_1, v_2 \in \mathbb{R}$ and $x, y \in \mathbb{R}^n$, as required.

For proving the surjectivity of Φ, let $p \in \widetilde{\mathcal{M}}$ be arbitrary and $\mathcal{N}_p$ be the leaf through p. Then $f|_{\mathcal{N}_p} \equiv c$ for some $c \in \mathbb{R}$. The point (c_0, v, x) with $c_0 := c - f(x^0)$ and $\exp_{\gamma(c_0)}(v\widetilde{V}(\gamma(c_0)) + \langle x, \vec{S}(c_0) \rangle) = p$ is then a preimage of p. □

It remains to verify that Φ is isometric, i.e. that $\Phi^*\widetilde{g} = g^H$ with g^H as in (2.6). For $k = 1, \dots, n$ let

$$\begin{aligned} \mathscr{V}(u,v,x) &:= d\Phi_{(u,v,x)}(\partial_v), \\ \mathscr{X}_k(u,v,x) &:= d\Phi_{(u,v,x)}(\partial_k), \\ \mathscr{Z}(u,v,x) &:= d\Phi_{(u,v,x)}(\partial_u) \end{aligned}$$

denote the push-forward vector fields. Then, since the leaves of $\widetilde{\mathbb{L}^{\perp}}$ are totally geodesic, we have that

$$\mathscr{V}(u,v,x) \in \widetilde{\mathbb{L}}^{\perp}_{\Phi(u,v,x)} \text{ and } \mathscr{X}_k(u,v,x) \in \widetilde{\mathbb{L}}^{\perp}_{\Phi(u,v,x)}$$

for $k = 1, \dots, n$. Furthermore, along the integral curve γ of $\widetilde{Z}$, we have $\mathscr{V}(u, 0, 0) = \widetilde{V}(\gamma(u))$ and $\mathscr{X}_k(u, 0, 0) = S_k(\gamma(u))$. This can be generalized to

Lemma 2.12. *For each $(u,v,x)\in\mathbb{R}^{n+2}$ the vector fields $t\mapsto\mathscr{V}(u,tv,tx)$ and $t\mapsto\mathscr{X}_k(u,tv,tx)$ are parallel transported along the geodesic $\mathbb{R}\ni t\longmapsto\delta(t):=\Phi(u,tv,tx)$.*

Proof. For each $(u,v,x)\in\mathbb{R}^{n+2}$ consider the geodesic variation $F:\mathbb{R}\times(-\varepsilon,\varepsilon)\longrightarrow\widetilde{\mathcal{M}}$,

$$F(t,s)=\Phi(u,t(v+s),tx)=\exp_{\gamma(u)}\Big(t\big((v+s)\widetilde{V}(\gamma(u))+\langle x,\vec{S}(u)\rangle\big)\Big)$$

of the geodesic $\delta(t)=F(t,0)$. The variation vector field along δ is given as

$$t\mapsto\frac{\partial F}{\partial s}(t,0)=dF|_{(t,0)}\left(\frac{\partial}{\partial s}\right)=d\Phi_{(u,tv,tx)}(t\partial_v)=t\,\mathscr{V}(u,tv,tx).$$

Thus, as the variation vector field of a variation of δ by geodesics, $Y(t):=t\mathscr{V}(u,tv,tx)$ is a Jacobi vector field along δ. Hence, since $\delta'(t)\in\widetilde{\mathbb{L}}^{\perp}_{\delta(t)}$ as well as $Y(t)\in\widetilde{\mathbb{L}}^{\perp}_{\delta(t)}$, we have

$$\frac{\nabla^2}{dt^2}Y(t)=R^{\widetilde{g}}(\delta'(t),Y(t))\delta(t)=0,$$

by the curvature properties of a pp-wave. We can apply Lemma 2.10 and obtain that $tV(u,tv,tx)$ is parallel transported along the geodesic δ. The same argument, using the geodesic variation

$$F_k(t,s):=\Phi(u,tv,tx+se_k),$$

shows that the $\mathscr{X}_k$ are parallel transported along δ. $\square$

Recall that for $t=0$ we know that $\mathscr{V}(u,0,0)=\widetilde{V}(\gamma(u))$ and $\mathscr{X}_k(u,0,0)=S_k(\gamma(u))$. On the one hand, since $\widetilde{V}$ is parallel, in particular along δ, this implies that $\mathscr{V}(u,tv,tx)=\widetilde{V}(\delta(t))$ and hence $\widetilde{V}=\mathscr{V}$ everywhere on $\widetilde{\mathcal{M}}$. On the other hand, it implies that

$$\begin{aligned}\widetilde{g}_{\phi(u,v,x)}(\mathscr{V}(u,v,x),\mathscr{V}(u,v,x)) &= \widetilde{g}_{\gamma(u)}(\widetilde{V}(\gamma(u)),\widetilde{V}(\gamma(u))) &= 0,\\ \widetilde{g}_{\phi(u,v,x)}(\mathscr{V}(u,v,x),\mathscr{X}_k(u,v,x)) &= \widetilde{g}_{\gamma(u)}(\widetilde{V}(\gamma(u)),S_k(\gamma(u))) &= 0,\\ \widetilde{g}_{\phi(u,v,x)}(\mathscr{X}_i(u,v,x),\mathscr{X}_j(u,v,x)) &= \widetilde{g}_{\gamma(u)}(S_i(\gamma(u)),S_j(\gamma(u))) &= \delta_{ij},\end{aligned}$$

and thus

$$\Phi^*\widetilde{g}(\partial_v,\partial_v)=0,\ \Phi^*\widetilde{g}(\partial_v,\partial_k)=0 \text{ and } \Phi^*\widetilde{g}(\partial_i,\partial_j)=\delta_{ij}.$$

It remains to show that $\Phi^*\widetilde{g}(\partial_k,\partial_u)=0$ and $\Phi^*\widetilde{g}(\partial_v,\partial_u)=1$. For the second equation consider for fixed $v\in\mathbb{R}$ and $x\in\mathbb{R}^n$ the variation

$$\nu(u,s):=\Phi(u,sv,sx)$$

and let $\nu_u(u,s):=\frac{\partial\nu}{\partial u}(u,s)$ and $\nu_s(u,s):=\frac{\partial\nu}{\partial s}(u,s)$. Observe that it holds $\nu_s\in\widetilde{\mathbb{L}}^{\perp}$ and $\mathscr{X}(u,sv,sx)=\nu_u(u,s)$. Consequently, by Schwarz' lemma and the parallelity of $\widetilde{\mathbb{L}}^{\perp}$,

$$\frac{\nabla}{ds}\nu_u(u,s)=\frac{\nabla}{du}\nu_s(u,s)\in\widetilde{\mathbb{L}}^{\perp}. \tag{2.11}$$

This implies

$$\frac{d}{ds}\widetilde{g}_{\nu(u,s)}(\nu_u(u,s),\widetilde{V}(\nu(u,s)))\equiv 0,$$

i.e., $s \mapsto \widetilde{g}(\nu_u(u,s), \widetilde{V}(\nu(u,s)))$ is constant and thus equals its value in $s = 0$, which is

$$\widetilde{g}_{\nu(u,0)}(\dot{\gamma}(u), \widetilde{V}(\gamma(u))) = \widetilde{g}_{\gamma(u)}(\widetilde{Z}(\gamma(u)), \widetilde{V}(\gamma(u))) \equiv 1,$$

since $\nu(u,0) = \gamma(u)$, $\nu_u(u,0) = \dot{\gamma}(u)$ and since γ is an integral curve of $\widetilde{Z}$. This proves $\Phi^*\widetilde{g}(\partial_v, \partial_u) = 1$.

To see $\Phi^*\widetilde{g}(\partial_k, \partial_u) = 0$, consider the identity

$$R^{\widetilde{g}}(\nu_u, \nu_s)\nu_s = \frac{\nabla}{du}\frac{\nabla}{ds}\nu_s - \frac{\nabla}{ds}\frac{\nabla}{du}\nu_s. \tag{2.12}$$

Since $s \mapsto \nu(u,s)$ is a geodesic for every $u \in \mathbb{R}$, it holds $\frac{\nabla}{ds}\nu_s = 0$. Taking into account that $\nu_s \in \widetilde{\mathbb{L}}^\perp$ we have by the definition of a pp-wave, see also Proposition 2.2(i.c), that $R^{\widetilde{g}}(\nu_u, \nu_s)\nu_s \in \mathbb{R} \cdot \widetilde{V}$ and hence (2.11) and (2.12) yield

$$\frac{\nabla}{ds}\frac{\nabla}{ds}\nu_u(u,s) = \varphi(\nu(u,s)) \cdot \widetilde{V}(\nu(u,s))$$

for some function $\varphi \in C^\infty(\widetilde{\mathcal{M}})$. We conclude that

$$\frac{d}{ds}\widetilde{g}_{\nu(u,s)}(\tfrac{\nabla}{ds}\nu_u(u,s), \mathscr{X}_k(u,sv,sx)) = \widetilde{g}_{\nu(u,s)}(\tfrac{\nabla}{ds}\nu_u(u,s), \tfrac{\nabla \mathscr{X}_k}{ds}(u,sv,sx)) = 0,$$

because of Lemma 2.12. Hence,

$$s \mapsto \widetilde{g}_{\nu(u,s)}(\tfrac{\nabla}{ds}\nu_u(u,s), \mathscr{X}_k(u,sv,sx))$$

is constant and equals its value in $s = 0$. But for $s = 0$ we have

$$\widetilde{g}_{\nu(u,0)}(\tfrac{\nabla}{ds}\nu_u(u,0), \mathscr{X}_k(u,0,0)) = 0,$$

since $\mathscr{X}_k(u,0,0) = S_k(\gamma(u))$, and

$$\begin{aligned}
\frac{\nabla}{ds}\nu_u(u,0) \;=\; \frac{\nabla}{du}\nu_s(u,0) \;&=\; \frac{\nabla}{du}\left(\frac{d}{ds}\left(\exp_{\gamma(u)}(s(v\widetilde{V}(\gamma(u)) + \langle x, \vec{S}(\gamma(u))\rangle)\right)\Big|_{s=0}\right)\\
&=\; \frac{\nabla}{du}\left(v \cdot \widetilde{V}(\gamma(u)) + \sum_{k=1}^n x^k S_k(\gamma(u))\right)\\
&=\; \sum_{k=1}^n x^k \alpha_k(\dot{\gamma}(u))\widetilde{V}(\gamma(u)),
\end{aligned}$$

by the Schwarz lemma, the parallelity of $\widetilde{V}$ and the property of S_k. Note that we use here that $\frac{\nabla}{du}(x^k S_k) = x^k \frac{\nabla}{du} S_k$ since the x^k are constant along the curve $\gamma(u)$. Hence,

$$\widetilde{g}_{\nu(u,s)}(\tfrac{\nabla}{ds}\nu_u(u,s), \mathscr{X}_k(u,sv,sx)) \equiv 0.$$

Finally, this and Lemma 2.12 imply that

$$\frac{d}{ds}\widetilde{g}_{\nu(u,s)}(\nu_u(u,s), \mathscr{X}_k(u,sv,sx)) = \widetilde{g}_{\nu(u,s)}(\tfrac{\nabla}{ds}\nu_u(u,s), \mathscr{X}_k(u,sv,sx)) = 0.$$

Hence, also $s \mapsto \widetilde{g}_{\nu(u,s)}(\nu_u(u,s), \mathscr{X}_k(u,sv,sx))$ is constant and as $\nu_u(u,0) = \dot{\gamma}(u)$ we obtain at $s = 0$:

$$\widetilde{g}_{\nu(u,0)}(\dot{\gamma}(u), S_k(\gamma(u))) = \widetilde{g}_{\gamma(t)}(\widetilde{Z}(\gamma(t)), S_k(\gamma(u))) = 0.$$

Thus, the only term in the metric $\Phi^*\widetilde{g}$ on $\mathbb{R}^{n+2}$ which is not constant, is the function $H \in C^\infty(\mathbb{R} \times \mathbb{R}^n)$ defined by

$$2H := (\Phi^*\widetilde{g})(\partial_u, \partial_u).$$

This finishes the proof of the second statement of Theorem 2.9. $\square$

Remark 2.13. *Note that, at this stage we do not make a claim about the geodesic completeness of pp-waves satisfying the assumptions of Theorem 2.9. This will depend on the function H. We will give a sufficient condition in Lemma 2.19 in the next section, which we will then use to establish completeness for compact pp-waves.*

In the proof of the first part of Theorem 2.9 we used the result by Palais that a simply-connected manifold which admits a parallelism consisting of complete vector fields with constant Lie brackets, i.e., constant linear combinations of these vector fields, admits a unique Lie group structure, for which the vector fields of the parallelism are left-invariant. For the sake of being self contained, we will prove directly the weaker statement that we need for our proof.

Lemma 2.14. *Let $\mathcal{N}$ be a manifold of dimension n and $X_1, \ldots, X_n$ complete vector fields.*

(i) If the X_i commute with each other, i.e., $[X_i, X_j] = 0$, then every vector field X that is a constant linear combination of the X_i's, i.e., $X = \sum_{i=1}^n a^i X_i$ with $a^i \in \mathbb{R}$, is complete.

(ii) If the X_i are linearly independent at each point in $\mathcal{N}$ and if h is a semi-Riemannian metric on $\mathcal{N}$ such that the X_i are parallel with respect to the Levi-Civita connection ∇^h, then $(\mathcal{N}, h)$ is geodesically complete. In particular, if $\mathcal{N}$ is simply-connected, then $(\mathcal{N}, h)$ is isometric to the standard semi-Euclidean metric on $\mathbb{R}^n$.

Proof. *Ad* (i). Let $X = \sum_{i=1}^n a^i X_i$ with $a^i \in \mathbb{R}$ and $\phi^i : \mathbb{R} \times \mathcal{N} \ni (t,p) \to \phi^i_t(p) \in \mathcal{N}$ be the flow of X_i. Let $p \in \mathcal{N}$ fixed. Then we claim that the map

$$\phi : \mathbb{R} \times \mathcal{N} \ni (t,p) \mapsto \phi^1_{a^1 t} \circ \ldots \circ \phi^n_{a^n t}(p) \in \mathcal{N}.$$

is the flow of X through p. Indeed, if we define

$$g : \mathbb{R} \ni t \longmapsto t \sum_{i=1}^n a^i e_i \in \mathbb{R}^n$$

where e_i is the standard basis of $\mathbb{R}^n$ and

$$f : \mathbb{R}^n \ni \sum_{i=1}^n a^i e_i \longmapsto \phi^1_{a^1} \circ \ldots \circ \phi^n_{a^n}(p) \in \mathcal{N}$$

then we have that $\phi = f \circ g$. For the differential of g we clearly have $dg_t(\partial_t) \equiv \sum_{i=1}^n a^i e_i$. Since the X_i's commute with each other, their flows commute, i.e., $\phi^i_t \circ \phi^j_s = \phi^j_s \circ \phi^i_t$, which implies that the differential of f is given as

$$df_{\sum_{i=1}^n a^i e_i}(e_j) = \frac{d}{dt}\left(\phi^j_t(\phi^1_{a^1} \circ \ldots \circ \phi^n_{a^n}(p))\right)\Big|_{t=0} = X_j(\phi^1_{a^1} \circ \ldots \circ \phi^n_{a^n}(p)).$$

Hence, by the chain rule we obtain

$$\frac{d}{dt}\phi_t(p) = d(f \circ g)_t(\partial_t) = df_{g(t)} dg_t(\partial_t) = \sum_{i=1}^n a^i df_{g(t)}(e_i) = \sum_{i=1}^n a^i X_i(\phi_t(p)) = X(\phi_t(p))$$

as claimed.

Ad (ii). Clearly, if the X_i are parallel with respect to ∇^h, they commute with each other and we can apply *(i)*. Now, let $w := \sum_{i=1}^n a^i X_i(p) \in T_p\mathcal{N}$ be an arbitrary tangent vector at an arbitrary point $p \in \mathcal{N}$. Then the vector field $X = \sum_{i=1}^n a^i X_i$ is parallel and, by *(i)*, complete. Hence, the geodesic starting at p with initial speed w is given by the flow of X through p, and thus defined on all of $\mathbb{R}$.

Finally, if $(\mathcal{N}, h)$ admits n parallel complete vector fields, it is flat and geodesically complete, and thus, with the assumption that $\mathcal{N}$ is simply-connected, the Killing-Hopf theorem gives us that $(\mathcal{N}, h)$ is isometric to the standard Euclidean vector space of dimension n. □

Theorem 2.9 provides us with the possibility to describe the universal cover for certain pp-waves that fulfill the required completeness assumptions made in the theorem. Indeed, for *compact* pp-waves these assumptions are always satisfied.

Theorem 2.15. *For a compact pp-wave the maximal geodesics along the leaves of the parallel distribution $\mathbb{L}^\perp$ are defined on $\mathbb{R}$.*

Proof. Again, by a tilde we shall denote the lift of an object to the universal cover. With Z we denote a complete screen vector field and with $\mathbb{S}$ the corresponding screen distribution on $\mathcal{M}$.

As $\mathcal{M}$ is compact, we can apply Proposition 1.46 and obtain that the universal cover $\widetilde{\mathcal{M}}$ of $\mathcal{M}$ is diffeomorphic to $\mathbb{R} \times \widetilde{\mathcal{L}^\perp}$, where $\widetilde{\mathcal{L}^\perp}$ is the universal cover of a leaf $\mathcal{L}^\perp$ of the distribution $\mathbb{L}^\perp$ of $\mathcal{M}$. Note that $\widetilde{\mathcal{L}^\perp}$ is also a leaf of the distribution $\widetilde{\mathbb{L}^\perp}$ on $\widetilde{\mathcal{M}}$. Since $(\mathcal{M}, g)$ is a pp-wave, the lift $\widetilde{\mathbb{S}}$ of the screen distribution $\mathbb{S}$ comes with a global frame field $S_i \in \Gamma(\widetilde{\mathbb{S}})$, $i = 1, \ldots, n$, on $\widetilde{\mathcal{M}}$ satisfying the relations

$$\nabla^{\widetilde{g}} S_i = \alpha^i \otimes \widetilde{V}, \tag{2.13}$$

see Proposition 2.4. Note that the S_i are not necessarily lifts of global vector fields on the compact $\mathcal{M}$, however we will show that they are complete.

To this end, consider the associated Riemannian metric h on $\mathcal{M}$ defined by the original screen distribution $\mathbb{S}$ on $\mathcal{M}$ via (1.19) on page 20. As a Riemannian metric on a compact manifold $\mathcal{M}$ it is geodesically complete, and so is its restriction to the leaves $\mathcal{L}^\perp$ by Proposition 1.26. Therefore, the lifted Riemannian metric $\widetilde{h}$ on $\widetilde{\mathcal{L}^\perp}$ is geodesically complete. Now one computes that the vector fields $S_1, \ldots, S_n$ on $\widetilde{\mathcal{L}^\perp}$, which are $\widetilde{h}$-orthonormal, span the lifted screen $\widetilde{\mathbb{S}}$ and satisfy equation (2.13), are in fact geodesic vector fields for $(\widetilde{\mathcal{L}^\perp}, \widetilde{h})$. Indeed, from the Koszul formula we get

$$0 = \widetilde{g}(\nabla^{\widetilde{g}}{}_{S_i} S_i, X) = S_i(\widetilde{g}(S_i, X)) + \widetilde{g}([X, S_i], S_i) = S_i(\widetilde{h}(S_i, X)) + \widetilde{h}([X, S_i], S_i) = \widetilde{h}(\nabla^{\widetilde{h}}_{S_i} S_i, X)$$

for all $X \in \Gamma(T\widetilde{\mathcal{L}^\perp})$. Here the replacement of $\widetilde{g}$ by $\widetilde{h}$ is justified since

$$\widetilde{g}(S_i, \cdot)|_{T\widetilde{\mathcal{L}^\perp}} = \widetilde{h}(S_i, \cdot)|_{T\widetilde{\mathcal{L}^\perp}}.$$

With $(\widetilde{\mathcal{L}^\perp}, \widetilde{h})$ being geodesically complete and S_i being geodesic vector fields, this yields the conclusion that the S_i are complete vector fields.

From $\widetilde{\mathbb{S}}$ and using Theorem 2.5 resp. Corollary 2.7, we obtain an involutive and horizontal screen distribution $\widehat{\mathbb{S}}$ which is spanned by vector fields

$$\widehat{S}_i = S_i - b^i \widetilde{V} \tag{2.14}$$

for some smooth function b^i on $\widetilde{\mathcal{M}}$ and which satisfy $\nabla^{\widetilde{g}} \widehat{S}_i|_{T\widetilde{\mathcal{L}^\perp}} = 0$. Let $\widehat{h}$ the associated Riemannian metric to $\widehat{\mathbb{S}}$ on $\widetilde{\mathcal{L}^\perp}$. Using Proposition 1.47 we finally obtain:

Lemma 2.16. *The vector fields $\widehat{S}_i$ are complete.*

Proof. We saw that the vector fields S_i on $\widetilde{\mathcal{L}^\perp}$ are complete, i.e., we obtain their flows as

$$\phi^i : \mathbb{R} \times \widetilde{\mathcal{L}^\perp} \longrightarrow \widetilde{\mathcal{L}^\perp}.$$

Recall that, by Proposition 1.47, the leaf $\widetilde{\mathcal{L}^\perp}$ is diffeomorphic to $\mathbb{R} \times \widehat{\mathcal{S}}$ via

$$\Psi : p \in \widetilde{\mathcal{L}^\perp} \longmapsto (\varphi(p), \psi_{-\varphi(p)}(p)) \in \mathbb{R} \times \widehat{\mathcal{S}},$$

were $\{\psi_t\}_{t\in\mathbb{R}}$ is the flow of $\widetilde{V}$ and $\varphi \in C^\infty(\widetilde{\mathcal{L}^\perp})$, such that $\widehat{h}(\widetilde{V}, \cdot)|_{\widetilde{\mathbb{L}}^\perp} = d\varphi$ (see the proof of Proposition 1.46). Under this diffeomorphism, the flows $\{\phi^i_t\}_{t\in\mathbb{R}}$ are given as

$$\widetilde{\phi}^i_t(p) := \Psi(\phi^i_t(p)) = (\nu^i_t(p), \widehat{\phi}^i_t(p)),$$

with

$$\begin{aligned} \nu^i_t(p) &:= \varphi(\phi^i_t(p)), \\ \widehat{\phi}^i_t(p) &:= \psi_{-\nu^i_t(p)}(\phi^i_t(p)), \end{aligned}$$

both defined for all $t \in \mathbb{R}$. We do now claim that $\{\widehat{\phi}^i_t\}_{t\in\mathbb{R}}$ is the flow of $\widehat{S}_i$. Indeed, on the one hand we have that

$$\frac{d}{dt}\widetilde{\phi}^i_t(p) = (\tfrac{d}{dt}\nu^i_t(p), \tfrac{d}{dt}\widehat{\phi}^i_t(p)) = (\tfrac{d}{dt}\nu^i_t(p), 0) + (0, \tfrac{d}{dt}\widehat{\phi}^i_t(p)). \tag{2.15}$$

On the other hand we compute, using (2.14), the chain rule and the linearity of the differential

$$\tfrac{d}{dt}\widetilde{\phi}^i_t(p) = d\Psi_{\phi^i_t(p)}(S_i(\phi^i_t(p))) = d\Psi_{\phi^i_t(p)}(\widehat{S}_i(\phi^i_t(p))) + b^i(\phi^i_t(p)) d\Psi_{\phi^i_t(p)}(\widetilde{V}(\phi^i_t(p))). \tag{2.16}$$

Temporarily denoting by $\{\xi^i_t\}$ the flow of $\widehat{S}_i$, for the first term we get

$$\begin{aligned} d\Psi_{\phi^i_t(p)}(\widehat{S}_i(\phi^i_t(p))) = \tfrac{d}{d\tau}\Psi(\xi^i_\tau(\phi^i_t(p)))\Big|_{\tau=0} &= \tfrac{d}{d\tau}\Big(\nu_t(p), \psi_{-\nu_t(p)} \circ \xi^i_\tau(\phi^i_t(p))\Big)\Big|_{\tau=0} \\ &= \tfrac{d}{d\tau}\Big(\nu_t(p), \xi^i_\tau \circ \psi_{-\nu_t(p)}(\phi^i_t(p))\Big)\Big|_{\tau=0} \\ &= \tfrac{d}{d\tau}\Big(\nu_t(p), \xi^i_\tau\Big(\widehat{\phi}^i_t(p)\Big)\Big)\Big|_{\tau=0} \\ &= (0, \widehat{S}_i(\widehat{\phi}^i_t)) \in \mathbb{R} \oplus T\widehat{\mathcal{S}}, \end{aligned}$$

in which we were allowed to commute the flows ξ^i_τ and ψ^i_s because of $[\widetilde{V}, \widehat{S}_i] = 0$. For the second term in (2.16) we recall that

$$\tfrac{d}{d\tau}\varphi(\psi_\tau(\phi^i_t(p)))|_{\tau=0} = d\varphi_{\phi^i_t(p)}(\widetilde{V}|_{\phi^i_t(p)}) = \widehat{h}(\widetilde{V}, \widetilde{V})|_{\phi^i_t(p)} \equiv 1,$$

which implies $\varphi(\psi_\tau(\phi_t^i(p)) = \tau + c$ for a constant c. Hence, we get

$$\begin{aligned} d\Psi_{\phi_t^i(p)}(\widetilde{V}(\phi_t^i(p))) &= \tfrac{d}{d\tau}\Psi(\psi_\tau(\phi_t^i(p)))\Big|_{\tau=0} \\ &= \tfrac{d}{d\tau}\Big(\varphi(\psi_\tau(\phi_t^i(p))), \psi_{-\varphi(\psi_\tau(\phi_t^i(p)))}(\psi_\tau(\phi_t^i(p)))\Big)\Big|_{\tau=0} \\ &= \tfrac{d}{d\tau}\Big(\tau + c, \psi_{-c}\circ\psi_{-\tau}(\psi_\tau(\phi_t^i(p)))\Big)\Big|_{\tau=0} \\ &= (1,0) \in \mathbb{R}\oplus T\widehat{\mathcal{S}} \end{aligned}$$

Both computations show that (2.16) becomes

$$\tfrac{d}{dt}\widetilde{\phi}_t^i(p) = \Big(b^i(\phi_t^i(p)), \widehat{S}_i(\widehat{\phi}_t^i)\Big) \in \mathbb{R}\oplus T\widehat{\mathcal{S}},$$

which, together with (2.15), shows that $\frac{d}{dt}\widehat{\phi}_t^i(p) = \widehat{S}_i(\widehat{\phi}_t^i(p))$. Hence, $\widehat{\phi}_t^i$ is the flow of $\widehat{S}_i$ which is defined on $\mathbb{R}$. This proves the lemma. □

The statement of the theorem is no immediate. Namely, $\nabla^{\widetilde{g}}|_{T\widetilde{\mathcal{L}}^\perp}$ and $\nabla^{\widehat{h}}$ coincide by Corollary 2.6, while $\nabla^{\widehat{h}}$ is complete by Lemma 2.16 and Proposition 1.22 together with Lemma 2.14(ii). Hence, $\nabla^{\widetilde{g}}|_{T\widetilde{\mathcal{L}}^\perp}$ is complete and consequently, so is $\nabla^g|_{T\mathcal{L}^\perp}$. □

Combining Theorem 2.9 and Theorem 2.15 we finally obtain the following description of the universal cover of compact pp-waves.

Theorem 2.17. *The universal cover of an $(n+2)$-dimensional compact pp-wave is globally isometric to a standard pp-wave*

$$\Big(\mathbb{R}^{n+2},\ \ g^H = 2dudv + 2H(u,x^1,\dots,x^n)du^2 + \delta_{ij}dx^idx^j\Big).$$

Under this isometry, the lift of the parallel light-like vector field is mapped to ∂_v.

2.3 COMPLETENESS

Using the results of the previous section we can investigate geodesic completeness of compact pp-waves by applying existing results for geodesic completeness of non-compact Lorentzian manifolds with parallel light-like vector field.

To our best knowledge, the strongest of such results can be found in [CFS03] and they hold for a special class of these manifolds described in the next theorem.

Theorem 2.18 ([CFS03, Theorem 3.2 and Corollary 3.4])**.** *Let $(\mathcal{S},h)$ be a connected Riemannian manifold of dimension n and let $H \in C^\infty(\mathbb{R}\times\mathcal{S})$ be a smooth function. On the manifold $\mathcal{M} := \mathbb{R}^2\times\mathcal{S}$ define the Lorentzian metric g by*

$$g_{(u,v,x)} = 2dudv + 2H(u,x)du^2 + h_x, \tag{2.17}$$

where $x\in\mathcal{S}$ and (u,v) are the global coordinates on $\mathbb{R}^2$.

(i) The Lorentzian manifold $(\mathcal{M}, g)$ is geodesically complete if and only if the Riemannian manifold $(\mathcal{S}, h)$ is complete and the solutions $s \mapsto \gamma(s)$ of the ODE

$$\frac{\nabla^h \dot{\gamma}}{ds}(s) = \operatorname{grad}_h H(s, \gamma(s)) \tag{2.18}$$

are defined on the whole real line. Here ∇^h is the Levi-Civita connection of h.

(ii) If $(\mathcal{S}, h)$ is geodesically complete and the function H does not depend on u and is at most quadratic at spacial infinity, i.e., there exist $x_0 \in \mathcal{S}$ and real constants $r, c > 0$ such that

$$H(x) \leq c \cdot d_{\mathcal{S}}(x_0, x)^2 \text{ for all } x \in \mathcal{S} \text{ with } d_{\mathcal{S}}(x_0, x) \geq r,$$

then $(\mathcal{M}, g)$ is geodesically complete. Here $d_{\mathcal{S}}$ is the distance function of $(\mathcal{S}, h)$.

This theorem applies to pp-waves in standard form (2.2), and to the more general class of pp-waves that are globally of the form (2.17) with $(\mathcal{S}, h)$ a flat Riemannian manifold, not necessarily the $\mathbb{R}^n$. As a corollary, the following lemma[3] provides a sufficient condition for the function H appearing in g^H of (2.2) to yield completeness of a standard pp-wave.

Lemma 2.19. *The pp-wave metric on $\mathbb{R}^{n+2}$ in standard form*

$$g^H = 2dudv + 2H(u, x^1, \ldots, x^n)du^2 + \delta_{ij}dx^i dx^j$$

is geodesically complete if all second x^i-derivatives of H are bounded, i.e. $\left|\frac{\partial^2 H}{\partial x^i \partial x^j}\right| \leq c$ for a positive constant c and $1 \leq i, j \leq n$.

Proof. By Theorem 2.18, g^H is complete if every maximal solution $\gamma : s \mapsto \gamma(s) \in \mathbb{R}^n$ of

$$\ddot{\gamma}(s) = F(s, \gamma) := \operatorname{grad}_{\mathbb{R}^n} H(s, \gamma(s)) \tag{2.19}$$

is defined on the whole real line. Now, recall the following fact, see for example [Tes12, Theorem 2.17]: *Let $F : \mathbb{R} \times \mathbb{R}^{2n} \to \mathbb{R}^n$ be globally Lipschitz on every set of the form $I \times \mathbb{R}^{2n}$, where I is a closed interval, then, for every initial value $(t_0, x_0, x_1) \in \mathbb{R} \times \mathbb{R}^{2n}$ there is a solution $x : \mathbb{R} \to \mathbb{R}^n$ of the initial value problem $\ddot{x} = F(t, x, \dot{x})$ with $x(t_0) = x_0$ and $\dot{x}(t_0) = x_1$.* We thus have to show, that the function $F : [a, b] \times \mathbb{R}^{2n} \longrightarrow \mathbb{R}^n$ with $F(s, x, y) = F(s, x)$ defined in (2.19) is Lipschitz for arbitrary $a, b \in \mathbb{R}$. Clearly, by the mean value theorem for functions from $\mathbb{R}^n$ to $\mathbb{R}^n$, if every partial derivative of F is bounded, then F is Lipschitz. But every partial derivative in the second argument of $F = (F_1, \ldots, F_n)$ is given by

$$\frac{\partial F_i}{\partial x_j}(t, x) = \frac{\partial}{\partial x_j}\left(\frac{\partial H}{\partial x_i}\right)(t, x),$$

and thus bounded by assumption. We conclude that F must be Lipschitz on every set $[a, b] \times \mathbb{R}^n$ which guarantees that the maximal solutions γ of (2.19) are defined on $\mathbb{R}$. □

3 In fact, during the preparation of the paper we learned that Lemma 2.19 follows from stronger results by Candela et al. [CRS13, Theorems 1 and 2]. However, for the sake of being self-contained we include a proof of the lemma. For further results and comments see [CRS12, Sán13].

As it turns out, for the universal cover of compact pp-waves the sufficient condition of the previous lemma is always satisfied.

Lemma 2.20. *Let $(\mathcal{M}^{(n+2)}, g)$ be a compact pp-wave and let $g^H = 2dudv + 2Hdu^2 + \delta_{ij}dx^i dx^j$ be the metric on the universal cover $\mathbb{R}^{n+2}$ of $\mathcal{M}$ that is globally isometric to the lift of g. Then all second covariant derivatives of H in x^i-directions are bounded,*

$$0 \leq \partial_i \partial_j H \leq c, \text{ for all } i, j = 1, \dots n.$$

Proof. Let $\phi : (\mathbb{R}^{n+2}, g^H) \longrightarrow (\mathcal{M}, g)$ denote the isometric universal covering map from Theorem 2.17. Let $Z \in \Gamma(T\mathcal{M})$ be an arbitrarily chosen screen vector field and $\widetilde{Z} \in \Gamma(T\widetilde{\mathcal{M}})$ its pullback to $\widetilde{\mathcal{M}}$. Note that we have particularly shown in Theorem 2.17 that $g(d\phi(\partial_u), V) = 1$, and hence we have that

$$d\phi(\partial_u) = Z + \sum_{i=1}^{n} b^i S_i + cV \tag{2.20}$$

for smooth functions $b^i, c \in C^\infty(\widetilde{\mathcal{M}})$ and S_i a basis of the screen distribution corresponding to Z. Now we define a symmetric $(2,0)$-tensor field on $\mathcal{M}$ as

$$Q(X, Y) := R^g(X, Z, Z, Y).$$

Since $\mathcal{M}$ is compact, the function $\overline{g}(Q, Q)$, where $\overline{g}$ denotes the metric induced by g on $(2,0)$-tensor fields, is bounded, i.e., $-C^2 < \overline{g}(Q, Q) < C^2$ for some constant $C \in \mathbb{R}^+$. Computing $\overline{g}(Q, Q)$ in a frame $V, Z, E_1, \dots, E_n$ with E_i an orthonormal frame of the screen defined by Z, the obvious equation $Q(V, .) = 0$ gives us

$$\overline{g}(Q, Q) \;=\; \sum_{i,j=1}^{n} Q(E_i, E_j)^2 \;=\; \sum_{i,j=1}^{n} R^g(E_i, Z, Z, E_j)^2,$$

so we have in fact that $0 \leq \overline{g}(Q, Q) < C^2$.

Pulling back Q to the universal cover $(\mathbb{R}^{n+2}, g^H)$ by the isometric covering map ϕ, using (2.20), (2.3) and (2.1), we get that $\phi^* Q(\partial_v, .) = 0$ and

$$\begin{aligned}
\phi^* Q(\partial_i, \partial_j)_x &= R^g_{\phi(x)}(d\phi_x(\partial_i), Z, Z, d\phi_x(\partial_j)) \\
&= R^g_{\phi(x)}(d\phi_x(\partial_i), d\phi_x(\partial_u), d\phi_x(\partial_u), d\phi_x(\partial_j)) \\
&= \phi^* R^{g^H}_x(\partial_i, \partial_u, \partial_u, \partial_j) \\
&= R^{g^H}_x(\partial_i, \partial_u, \partial_u, \partial_j) \\
&= -\partial_i \partial_j H(x).
\end{aligned}$$

Hence, by using a frame $(\partial_v, \partial_u - H\partial_v, \partial_i)$ on $(\mathbb{R}^{n+2}, g^H)$ to compute $\overline{g^H}(\phi^* Q, \phi^* Q)$, at each point in $\mathbb{R}^{n+2}$ we have

$$C^2 > \overline{g}(Q, Q) = \overline{g^H}(\phi^* Q, \phi^* Q) = \sum_{i,j=1}^{n} \phi^* Q(\partial_i, \partial_j)^2 = \sum_{i,j=1}^{n} (\partial_i \partial_j H)^2,$$

which shows that all $\partial_i \partial_j H$ are bounded. □

Altogether, we deduce from Theorem 2.17, Lemma 2.19 and Lemma 2.20 the completeness of compact pp-waves.

Theorem 2.21. *Every compact pp-wave is geodesically complete.*

We stress that attempting to generalize the results in this chapter to compact *decent* Lorentzian manifolds, is not straightforward. Namely, the following example provides a compact decent Lorentzian manifold which admits no parallel vector field but with the curvature condition of a pp-wave[4] that is *not* complete (see also results by SÁNCHEZ [Sán97] on (incomplete) Lorentzian 2-tori).

Example 2.22. *Consider $\mathbb{R}^{n+2}$ endowed with the metric*

$$\widetilde{g}_{(u,v,x_1,\ldots,x_n)} := 2dudv - 2H(v,x_1,\ldots,x_n)du^2 + \sum_{i=1}^{n} dx_i^2$$

for

$$H(v,x_1,\ldots,x_n) := \sin(v) - \sum_{i=1}^{n} a_i(\cos(x_i) - 1)$$

for constants $a_i \in \mathbb{R}$ not all zero. Being 2π-periodic, the metric $\widetilde{g}$ descends to a metric g on the torus $\mathbb{T}^{n+2} := \mathbb{R}^{n+2}/2\pi\mathbb{Z}^{n+2}$. The inextensible (transversal) geodesic $\widetilde{\gamma}(t) := (\ln(t), 0, \ldots, 0)$ then defines an inextensible geodesic $\gamma : (0,\infty) \longrightarrow \mathbb{T}^{n+2}$ on the compact Lorentzian manifold $(\mathbb{T}^{n+2}, g)$ by $\gamma(t) := \pi(\widetilde{\gamma}(t))$, with $\pi : \mathbb{R}^{n+2} \longrightarrow \mathbb{T}^{n+2}$ denoting the canonical projection. For all $a_i = 0$ this is a version of the Clifton-Pohl torus and the obtained Lorentzian manifold becomes reducible.

2.4 PLANE WAVES

Another important class of Lorentzian manifolds with special holonomy are the *plane waves* (see definition below), which are pp-waves on which another condition on the curvature tensor is imposed. As it was mentioned in the introduction, plane waves occur within in the context of general relativity as solutions of the Einstein field equations. However, *compact* plane waves are also interesting for mathematicians since, for example, the Lorentzian manifolds with essentially parallel Weyl tensor (i.e. which are neither locally symmetric nor conformally flat) are plane waves as was shown by DERDZIŃSKI and ROTER in [DR09]. We will first make precise how plane waves are defined, apply our results to study their universal covering and then explain how our result can be applied to solve the Ehlers-Kundt Problem in the compact case.

Plane Waves

Definition 2.23. *A pp-wave $(\mathcal{M}^{(n+2)}, g)$ with parallel light-like vector field $V \in \Gamma(T\mathcal{M})$ is called a* **plane wave** *if for some $(4,0)$-tensor field Q its curvature tensor R^g satisfies*

$$\nabla^g R^g = V^\flat \otimes Q. \tag{2.21}$$

4 In [Lei06] these were called *pr-waves* for *plane fronted with recurrent rays*.

For a plane wave, the function H in the local form (2.2) of a pp-wave metric is quadratic in the x^i-coordinates

$$H(u, x^1, \dots x^n) = a_{ij}(u) x^i x^j, \text{ with } a_{ij} = a_{ji} \in C^\infty(\mathbb{R}). \tag{2.22}$$

This can be used to show that *standard plane waves* $(\mathbb{R}^{n+2}, g^H)$ with H as in (2.22) are always geodesically complete.

Proposition 2.24 ([CFS03, Proposition 3.5])**.** *Let $(\mathcal{M}^{(n+2)}, g)$ be a Lorentzian manifold of the form* (2.17) *and assume that its universal cover is globally isometric to a standard plane-wave*

$$\left(\mathbb{R}^{n+2}, \widetilde{g} = 2dudv + 2a_{ij}(u)x^i x^j du^2 + \delta_{ij} dx^i dx^j\right),$$

with $a_{ij} = a_{ji} \in C^\infty(\mathbb{R})$. Then $(\mathcal{M}, g)$ is geodesically complete.

In fact, this is immediate by Theorem 2.18 since in this case equation (2.18) is a *linear* second order ODE, whence their solutions are always defined on all of $\mathbb{R}$.

Using Theorem 2.17 together with Theorem 2.21 we get the following result about the universal cover of plane waves.

Corollary 2.25. *An $(n+2)$-dimensional compact plane wave is geodesically complete and its universal cover is isometric to $\mathbb{R}^{n+2}$ with the metric g^H defined in Theorem 2.17, where $H(u,x) = \sum_{k,l=1}^n a_{kl}(u) x^k x^l$ for some $a_{kl} = a_{lk} \in C^\infty(\mathbb{R})$.*

Proof. Since plane waves are pp-waves, Theorem 2.21 implies that compact plane waves are complete. Furthermore, by Theorem 2.17, we have for the universal covering that

$$R^{g^H}(\partial_i, \partial_u, \partial_u, \partial_j) = (\text{Hess } H)(\partial_i, \partial_j) = \partial_i(\partial_j(H)).$$

The additional plane wave condition $\nabla^g R^g = V^\flat \otimes Q$ implies for the universal cover that

$$0 = (\nabla^{g^H}_{\partial_k} R^{g^H})(\partial_i, \partial_u, \partial_u, \partial_j) = -\partial_k \partial_i \partial_j (H),$$

since $\nabla^{g^H}_{\partial_k} \partial_u \in \partial_v^\perp$. This implies that

$$H(u,x) = \sum_{k,l=1}^n a_{kl}(u) x^k x^l + \sum_{k=1}^n b_k(u) x^k + c(u).$$

Getting rid of the linear and constant terms in this expression is achieved by a coordinate transformation of the form

$$(\widetilde{v} = v - \dot{\beta}_i(u) x^i + \gamma(u),\ \widetilde{x}^i = x^i + \beta_i(u),\ \widetilde{u} = u)$$

where β and γ are obtained by integrating

$$\begin{aligned} \ddot{\beta}_i(u) &= -b_i(u), \\ \dot{\gamma}(u) &= c(u) - \tfrac{1}{2} \sum_{i=1}^n \dot{\beta}_i(u)^2, \end{aligned}$$

with initial conditions $\beta_i(0) = 0$ and $\gamma(0) = 0$. ☐

Connections to the Ehlers-Kundt Problem

Let us first restate the Ehlers-Kundt problem from the introduction (see page 6).

Problem (Ehlers-Kundt [EK62, Section 2-5.7])**.** *Prove the plane waves to be the only g-complete pp-waves, no matter which topology one chooses.*

A partial answer to this problem is given by the results in Proposition 2.24 and Theorem 2.18.

Proposition 2.26 ([FS06, Theorem 4])**.** *Any gravitational (Ricci-flat and four-dimensional) pp-wave (in standard form) such that H behaves at most quadratically at spatial infinity (in the sense of Theorem 2.18) is a (necessarily complete) plane wave.*

In contrast, the following provides examples for compact (and hence complete) pp-waves which are not plane waves.

Example 2.27. *Let η be the flat metric on the n-torus $\mathbb{T}^n$ and $H \in C^\infty(\mathbb{T}^n)$ a smooth function on $\mathbb{T}^n$. On $\mathcal{M} := \mathbb{T}^2 \times \mathbb{T}^n$ we consider the Lorentzian metric*

$$g^H = 2d\theta d\varphi + 2Hd\theta^2 + \eta, \tag{2.23}$$

where $d\theta$ and $d\varphi$ is the standard coframe on $\mathbb{T}^2$. This metric is a complete pp-wave metric on the torus $\mathbb{T}^{n+2}$, and one can choose H in a way that it is not a plane wave. Indeed, computing ∇R shows that for any function H with non-vanishing third partial derivatives with respect to the x^i-coordinates, the equality (2.21) *is violated. More examples are given in [Lär11, BLL14] and in Example 4.8.*

However, this example is not in contradiction to the claim in the Ehlers-Kundt problem because there, pp-waves are understood to be solutions of the Einstein *vacuum* field equations and hence, in addition to Definition 2.1, are assumed to be Ricci flat. But the metric (2.23) is Ricci flat if and only if H is harmonic with respect to the flat metric on the torus, which forces H to be constant and g^H to be flat. In fact, Theorem 2.17 and Lemma 2.20 allow us to generalize this observation.

Corollary 2.28. *Every compact Ricci-flat pp-wave is a plane wave.*

Proof. Let $(\mathcal{M}^{(n+2)}, g)$ be a compact pp-wave and let $(\mathbb{R}^{n+2}, g^H)$ be the standard pp-wave that is globally isometric to the universal cover of $(\mathcal{M}^{(n+2)}, g)$. Lemma 2.20 tells us that the $\partial_i\partial_j H$ are bounded. If g is Ricci-flat, so is g^H, and thus H is harmonic with respect to the x^i-directions by (2.4), i.e., $\Delta(H) = -\sum_{i,j=1}^n \partial_i\partial_j(H) = 0$. But this implies that also $F_{ij} := \partial_i\partial_j H$ is harmonic since

$$\Delta F_{ij} = \partial_i\partial_j(\Delta H) = 0$$

by the Schwarz lemma. Hence, by the maximum principle for harmonic functions, the F_{ij} are independent of the x^i components. Therefore,

$$H(u, x) = \sum_{i,j=1}^n a_{ij}(u)x^i x^j + b_i(u)x^i + c(u)$$

with a_{ij}, b_i and c functions of u only, which implies that $(\mathcal{M}, g)$ is a plane wave. $\square$

This solves the Ehlers-Kundt problem in case of compact manifolds. We stress that more, even non-compact, examples of complete pp-waves which are not plane waves can be obtained from the following.

Remark 2.29. *In regard to the Ehlers-Kundt problem, Lemma 2.19 provides us with many examples of pp-waves that are not plane waves. Again, these examples cannot be Ricci-flat, since harmonic functions do not have bounded second derivatives unless they are quadratic and thus a pp-wave.*

2.5 MANIFOLDS WITH ESSENTIALLY PARALLEL WEYL TENSOR

Let $(\mathcal{M}^{(n+2)}, g)$ a Lorentzian manifold of dimension $(n+2)$. Throughout this section we denote with

$$W^g = R^g + P^g \owedge g$$

the Weyl tensor of g, where $P^g := \frac{1}{n}(\mathrm{Ric}^g - \frac{\mathrm{scal}^g}{2(n+1)} g)$ is the *Schouten tensor* and $\owedge$ denotes the Kulkarni-Nomizu product. A Lorentzian manifold is said to have *essentially parallel Weyl tensor*[5] if and only if $\nabla^g W^g = 0$ but neither $W^g = 0$ nor $\nabla^g R^g = 0$. For a survey about recent results related to these manifolds, we refer to [DR07].

General Facts

Lorentzian manifolds $(\mathcal{M}^{(n+2)}, g)$ with essentially parallel Weyl tensor have special holonomy and the parallel line bundle $\mathbb{L}$ has an interesting description obtained via the aid of the Weyl tensor:

$$\mathbb{L} \cong \mathbb{D} := \bigsqcup_{x \in \mathcal{M}} \left\{ v \in T_x\mathcal{M} \mid g_x(v, \cdot) \wedge W_x^g(w, w', \cdot, \cdot) = 0 \ \forall w, w' \in T_x\mathcal{M} \right\},$$

where $\mathbb{D}$ is called the *Olszak distribution* [Ols93] of $(\mathcal{M}^{(n+2)}, g)$ cf. [DR09, Section 2]. Indeed, Derdziński and Roter have proven that every Lorentzian manifold with essentially parallel Weyl tensor is a plane wave [DR09]. Moreover, they proved a similar result to our Corollary 2.25 in [DR08, Theorem 7.1] but with an additional completeness assumption which can be omitted as our results have proven. We obtain:

Proposition 2.30. *Let $(\mathcal{M}^{(n+2)}, g)$ be a compact Lorentzian manifold with essentially parallel Weyl tensor. Then the universal cover is isometric to a manifold*

$$(\mathbb{R}^2 \times V, 2dtds + \kappa(t, s, \psi)dt^2 + \Theta), \tag{2.24}$$

where V is a real vector space of dimension n with an Euclidean inner product $\langle \cdot, \cdot \rangle$. Furthermore, t and s denote the Cartesian coordinates on the $\mathbb{R}^2$ factor, $\Theta := \pi^* \langle \cdot, \cdot \rangle$ for $\pi : \mathbb{R}^2 \times V \longrightarrow V$ and $\kappa : \mathbb{R}^2 \times V \longrightarrow V$ is defined by $\kappa(t, s, \psi) := f(t) \langle \psi, \psi \rangle + \langle A\psi, \psi \rangle$, where $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a periodic non-constant smooth function and $A \in \mathrm{End}(V)$ is a nonzero traceless operator, self-adjoint relative to $\langle \cdot, \cdot \rangle$.

5 We stress that usually these manifolds are referred to as *essentially conformally symmetric* (ECS). Unfortunately this term is a little misleading so we decided to use the term *essentially parallel Weyl tensor* as this describes the properties more carefully.

The Full Holonomy Group

We aim to describe the full holonomy of compact Lorentzian manifolds $(\mathcal{M}^{(n+2)}, g)$ with essentially parallel Weyl tensor. For this purpose we will apply Theorem 1.12 but are faced with the drawback that we need an explicit description of the fundamental group $\pi_1(\mathcal{M})$. Of course, in [DR10] there was introduced a group $G \subset \mathrm{Isom}(\widetilde{\mathcal{M}}, \widetilde{g})$ s.t. certain discrete subgroups $\Gamma \subset G$ yield compact examples of Lorentzian manifolds with essentially parallel Weyl tensor. More generally, if $\Gamma = \pi_1(\mathcal{M})$ is the fundamental group of a compact Lorentzian manifold with essentially parallel Weyl tensor, then there is a subgroup $H \subset \Gamma$ s.t. H has finite index in Γ and $H \subset G$ [DR08, Lemma 6.1 + Section 12].

The group G is constructed as follows. Let f, A, V and $\langle \cdot, \cdot \rangle$ denote the objects occurring in Proposition 2.30. Given the solution space

$$\mathscr{E} := \{u : \mathbb{R} \longrightarrow V \text{ smooth} \mid \ddot{u}(t) = f(t)u(t) + Au(t)\}, \tag{2.25}$$

the non-degenerate skew-symmetric bilinear form

$$\Omega(u_1, u_2) := \langle \dot{u}_1, u_2 \rangle - \langle u_1, \dot{u}_2 \rangle$$

which is constant for all $t \in \mathbb{R}$ since $A \in \mathrm{End}(V)$ is self-adjoint relative to $\langle \cdot, \cdot \rangle$, and the linear isomorphism $T : \mathscr{E} \longrightarrow \mathscr{E}$ with $(Tu)(t) := u(t-p)$ (where $p \in \mathbb{R}$ denotes the period of $f \in C^\infty(\mathbb{R})$), one defines

$$G := \mathbb{Z} \times \mathbb{R} \times \mathscr{E} \tag{2.26}$$

with $g_1 \cdot g_2$ for $g_i := (k_i, x_i, u_i)$ defined through the formula

$$g_1 \cdot g_2 := (k_1 + k_2, x_1 + x_2 - \Omega(u_1, T^{k_1} u_2), T^{-k_2} u_1 + u_2). \tag{2.27}$$

For $g = (k, x, u)$ and $m = (t, s, v) \in \widetilde{\mathcal{M}} = \mathbb{R} \times \mathbb{R} \times V$, the action of G on $\mathbb{R}^2 \times V$ then is given through

$$g \cdot m := (t + kp, s + x - \langle \dot{u}(t), 2v + u(t) \rangle, v + u(t)). \tag{2.28}$$

One easily verifies that each $F_g : m \in \widetilde{\mathcal{M}} \longrightarrow g \cdot m \in \widetilde{\mathcal{M}}$ is an isometry for the metric (2.24), i.e. $G \subset \mathrm{Isom}(\widetilde{\mathcal{M}}, \widetilde{g})$ as desired.

As the non-degenerate skew-symmetric bilinear form $\Omega \in \Lambda^2 \mathscr{E}^*$ is constant, the pair $(\mathscr{E}, \Omega)$ defines a symplectic vector space of dimension $2n$. To every such vector space we can associate a Heisenberg group $\mathrm{He}(\mathscr{E}, \Omega)$, cf. [Til70, Section I.3], if we endow $\mathrm{He}(\mathscr{E}, \Omega) := \mathbb{R} \times \mathscr{E}$ with the group structure $g_1 g_2 := (t_1 + t_2 + \Omega(u_1, u_2), u_1 + u_2)$ for $g_i = (t_i, u_i)$. By fixing a Darboux basis in $\mathscr{E}$, this group is isomorphic the canonical Heisenberg group $\mathrm{He}(n)$ which in matrix representation is defined as

$$\mathrm{He}(n) := \left\{ \begin{pmatrix} 1 & a^T & c \\ 0 & \mathbb{I} & b \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; a, b \in \mathbb{R}^n,\ c \in \mathbb{R} \right\}.$$

Moreover, we define for $B : \mathbb{R} \longrightarrow \mathrm{End}(V)$ with $\dot{B} + B^2 = f \cdot \mathbb{I} + A$ by

$$\mathscr{L} := \{u : \mathbb{R} \longrightarrow V \text{ smooth} \mid \dot{u}(t) = B(t)u(t)\} \subset \mathscr{E} \tag{2.29}$$

an n-dimensional subspace of $\mathscr{E}$.

In fact, each subgroup $\Gamma \subset G$ producing a compact quotient manifold $\widetilde{\mathcal{M}}/\Gamma$ implies the existence of a normal subgroup $\Sigma \subset \mathbb{R} \times \mathscr{L}$ which is a lattice and defined through $\Sigma := \Gamma \cap \ker \Pi$ with $\Pi : G \twoheadrightarrow \mathbb{Z}$ being the surjective homomorphism $\Pi(k, q, u) := k$, see [DR10, Section 4 + Theorem 6.1].

Using this, we can describe G and Γ more explicitly in terms of the groups $\mathbb{Z}$, Σ and $\mathrm{He}(n)$. Namely, since $\mathbb{Z}$ is a free group and

$$\phi : (t, u) \in \mathbb{R} \times \mathscr{E} \cong \{0\} \times \mathbb{R} \times \mathscr{E} \subset G \longmapsto (-t, u) \in \mathrm{He}(\mathscr{E}, \Omega)$$

is a Lie group isomorphism, the short exact sequences

$$0 \longrightarrow \mathbb{R} \times \mathscr{E} \cong_{\phi} \mathrm{He}(\mathscr{E}, \Omega) \cong \mathrm{He}(n-2) \xhookrightarrow{\iota} G \xrightarrow{\Pi} \mathbb{Z} \longrightarrow 0 \tag{2.30}$$

$$0 \longrightarrow \Sigma \xhookrightarrow{\iota} \Gamma \xrightarrow{\Pi} \mathbb{Z} \longrightarrow 0 \tag{2.31}$$

split. Hence:

Lemma 2.31. *For G and Σ as above it holds:*

(i) $G \cong \mathbb{Z} \ltimes \mathrm{He}(n)$.

(ii) $\Gamma \cong \mathbb{Z} \ltimes \Sigma$.

Given a compact Lorentzian manifold $(\mathcal{M}^{(n+2)}, g)$ with essentially parallel Weyl tensor, we see by Lemma 2.31 and (2.28) that every isometry $\sigma \in \pi_1(\mathcal{M}) \subset G \subset \mathrm{Isom}(\widetilde{\mathcal{M}}, \widetilde{g})$ acts only by translations. Therefore, Theorem 1.12 gives us the following.

Proposition 2.32. *Every compact Lorentzian manifold $(\mathcal{M}^{(n+2)}, g)$ with essentially parallel Weyl tensor whose fundamental group is contained in $\mathbb{Z} \ltimes \mathrm{He}(n)$ and thus isomorphic to $\mathbb{Z} \ltimes \Sigma$ for some lattice $\Sigma \subset \mathrm{He}(n)$ has full holonomy equal to $\mathbb{R}^n$.*

The Isometry Group

We aim to understand in this subsection how "strong" the restriction on the fundamental group made in Proposition 2.32 is. Indeed, if we still denote with $(\widetilde{\mathcal{M}}^{(n+2)}, \widetilde{g})$ the manifold $(\mathbb{R}^2 \times V, 2dtds + \kappa dt^2 + \Theta)$ as in Proposition 2.30, the fact that the group $\mathbb{R} \times \mathscr{E}$ as subgroup of $\mathrm{Isom}(\widetilde{\mathcal{M}}^{(n+2)}, \widetilde{g})$ is isomorphic to the n-dimensional Heisenberg group $\mathrm{He}(n)$, cf. Lemma 2.31, is not surprising. Namely, Blau et al. [BO03] proved that for pp-wave metrics (2.2) with $H(u, x) = \sum_{i=1}^{n} K(u) x_i^2$, the Lie algebra $\mathfrak{kill}(\widetilde{\mathcal{M}}^{(n+2)}, \widetilde{g})$ is equal to the Lie algebra $\mathfrak{he}(n)$ of $\mathrm{He}(n)$ if K does not fulfill special properties. As $\mathrm{He}(n)$ is simply-connected we infer for this case that $\mathrm{Isom}^0(\widetilde{\mathcal{M}}^{(n+2)}, \widetilde{g}) \cong \mathrm{He}(n)$. Therefore the restriction on the fundamental group $\pi_1(M) = \Gamma$ made in Proposition 2.32 turns out to be equivalent to requiring that $\pi_1(M)$ is contained in $\mathbb{Z} \ltimes \mathrm{Isom}^0(\widetilde{\mathcal{M}}^{(n+2)}, \widetilde{g})$. However, if the smooth function $\kappa \in \mathbb{R}^2 \times V \longrightarrow \mathbb{R}$ in (2.24) has additional properties, the identity component $\mathrm{Isom}^0(\widetilde{\mathcal{M}}^{(n+2)}, \widetilde{g})$ might be larger.

The present section should now give a complete answer to the question, how the identity component $\mathrm{Isom}^0(\widetilde{\mathcal{M}}^{(n+2)}, \widetilde{g})$ could look like. In particular we will see that the dimension d (as manifold) of $\mathrm{Isom}^0(\widetilde{\mathcal{M}}^{(n+2)}, \widetilde{g})$ (and consequently of $\mathrm{Isom}(\widetilde{\mathcal{M}}^{(n+2)}, \widetilde{g})$)

can be pretty large, i.e. we will prove that for particular endomorphisms $A \in \mathrm{End}(V)$ we have $d = \frac{(n+2)(n+3)}{2} - \dim \mathfrak{he}(n)$.

We begin with the asserted description of the identity component $\mathrm{Isom}^0(\widetilde{\mathcal{M}}^{(n+2)}, \widetilde{g})$.

Theorem 2.33. *Let $(\widetilde{\mathcal{M}}^{(n+2)}, \widetilde{g})$ denote a Lorentzian manifold with essentially parallel Weyl tensor as in Proposition 2.32. Then the identity component $\mathrm{Isom}^0(\widetilde{\mathcal{M}}^{(n+2)}, \widetilde{g})$ of the isometry group of $(\widetilde{\mathcal{M}}^{(n+2)}, \widetilde{g})$ is isomorphic to $\mathcal{S} \ltimes \mathrm{He}(n)$, where $\mathcal{S} \subset \mathrm{SO}(n)$ is a connected Lie subgroup of $\mathrm{SO}(n)$ with Lie algebra $\mathfrak{s} := \mathrm{span}\{F \in \mathfrak{so}(n) \mid [A, F] = 0\}$ which is non-trivial if and only if $A \in \mathrm{End}(V)$ has at least one eigenspace of dimension greater than one.*

Proof. In [BO03, Section 2.3] it is shown that there always exist $2n+1$ distinct Killing vector fields $E_1, \ldots, E_n, E_1^*, \ldots, E_n^*, Z$ for $\widetilde{g}$ which span $\mathfrak{he}(n)$. Note that in our notation the matrix $A(x^+)$ in [BO03] equals $(f(t) + \lambda_i)\delta_{ij}$ where λ_i, $i = 1, \ldots, n$, denote the eigenvalues of $A \in \mathrm{End}(V)$. Indeed, choosing an orthonormal frame in V consisting of eigenvectors $(v_1, \ldots, v_n)$ and $x_i(t,s,v) := r_i \in \mathbb{R}$, $v = \sum_i r_i v_i$, our $\widetilde{g}$ becomes

$$\widetilde{g} = 2dtds + \sum_{i=1}^{n} (f(t) + \lambda_i) x_i^2 dt^2 + \sum_{i=1}^{n} dx_i^2.$$

Concretely, these Killing vector fields $E_1, \ldots, E_n, E_1^*, \ldots, E_n^*, Z$ are given through

$$E_i = \textstyle\sum_{k=1}^{n} (\xi_{i,k}\partial_k - \dot{\xi}_{i,k} x_k \partial_s), \tag{2.32}$$

$$E_i^* = \textstyle\sum_{k=1}^{n} (\xi_{i,k}^*\partial_k - \dot{\xi}_{i,k}^* x_k \partial_s), \tag{2.33}$$

$$Z = \partial_s, \tag{2.34}$$

where $(\xi_1, \ldots, \xi_n, \xi_1^*, \ldots, \xi_n^*)$ is the basis of $\mathscr{E}$ with $\xi_{i,k}(0) = \delta_{ik}$, $\dot{\xi}_{i,k}(0) = 0$, $\xi_{i,k}^*(0) = 0$ and $\dot{\xi}_{i,k}^*(0) = \delta_{ik}$, cf. [BO03, Section 2.3].

However, there may exist additional Killing vector fields. Namely, this occurs in three distinct cases. The first case produces one additional Killing vector, iff the matrix $K(t) := (f(t) + \lambda_i)\delta_{ij}$ is degenerate for all $t \in \mathbb{R}$ which in our case obviously cannot occur as $f \in C^\infty(\mathbb{R})$ is non-constant. Next, the authors of [BO03] determine those cases in which they get additional Killing vector fields with ∂_t-component (which they refer to as *homogeneous* plane waves). As it turns out, these require $K(t)$ to be of the form $D(t) = \exp(tF) D_0 \exp(-tF)$ or $D(t) = \frac{1}{t^2} \cdot \exp(\ln(t)F) D_0 \exp(-\ln(t)F)$ for skew-symmetric F and symmetric D_0. But in our case, $D(t)$ cannot be of this form since $\mathrm{trace}\, K(t) = \mathrm{trace}(f(t) + \lambda_i)\delta_{ij} = nf(t)$ but $\mathrm{trace}\, D(t) = \mathrm{trace}\, D_0 \equiv \mathit{const}$ or $\mathrm{trace}\, D(t) = \frac{1}{t^2} \cdot \mathit{const}$ while f must be periodic.

The remaining case in which additional Killing vector fields can occur is the following. Considering for $F \in \mathfrak{so}(n)$ the equation

$$K(t) \cdot F - F \cdot K(t) \equiv 0, \tag{2.35}$$

it is proven in [BO03] that each such F generates an additional Killing vector field

$$X_F := \sum_{i,j=1}^{n} F_{ji} x^i \partial_j. \tag{2.36}$$

Writing down (2.35) componentwise one sees that $F_{ij} \neq 0$ requires $\lambda_i = \lambda_j$ for the eigenvalues $\lambda_i, \lambda_j \in \mathbb{R}$ of A. Thus, non-trivial solutions of (2.35) occur if and only if $A \in \mathrm{End}(V)$ has at least one eigenspace of dimension greater than one. Taking into account the generated Killing fields (2.36) we obtain for each such $F \in \mathfrak{so}(n)$ solving (2.35) a one-parameter subgroup of $\mathrm{SO}(n)$. Namely, the integral curves $\gamma_x^F : \mathbb{R} \longrightarrow V$ through $x \in V$ of such an X_F satisfy the differential equation $\dot{\gamma}_x^F(t) = F \cdot \gamma_x^F(t)$ with $\gamma(0) = x$ whose solution is given through $\gamma(t) = \exp(t \cdot F) \cdot x$ defined on the whole real line. Thus every Killing vector field X_F gives rise to a one-parameter subgroup $\{\Phi_\tau^F\}_{\tau \in \mathbb{R}}$ of $\mathrm{Isom}(\widetilde{\mathcal{M}}^{(n+2)}, \widetilde{g})$ via

$$\Phi_\tau : (t, s, v) \in \widetilde{\mathcal{M}} = \mathbb{R}^2 \times V \longmapsto (t, s, \exp(\tau F)v) \in \widetilde{\mathcal{M}}.$$

Indeed, every $\exp(\tau F) \in \mathrm{SO}(n)$ commutes with A as can be seen by differentiating the equation (which holds since $AF = FA$)

$$\exp(sA)\exp(\tau F) = \exp(sA + \tau F) = \exp(\tau F)\exp(sA)$$

in $s = 0$. Therefore, $\kappa \circ \Phi_\tau = \kappa$ and each Φ_τ is an isometry for $\widetilde{g}$. The one-parameter groups now generate the Lie group $\mathcal{S}$ stated in the theorem. Taking into account the commutator relations of elements in $\mathfrak{he}(n)$ with elements in $\mathfrak{s}$, namely

$$[X_F, E_i] = \sum_{\ell=1}^{n} F_{i\ell} E_\ell, \quad [X_F, E_i^*] = \sum_{\ell=1}^{n} F_{i\ell} E_\ell^* \text{ and } [X_F, Z] = 0,$$

cf. (2.32) – (2.34) and (2.36), and considering the homomorphism $\pi : \mathcal{S} \longrightarrow \mathrm{Aut}(\mathrm{He}(n))$ defined through

$$\pi(\exp(F)) \begin{pmatrix} 1 & a^T & c \\ 0 & \mathbb{I} & b \\ 0 & 0 & 1 \end{pmatrix} := \begin{pmatrix} 1 & (e^F a)^T & c \\ 0 & \mathbb{I} & e^F b \\ 0 & 0 & 1 \end{pmatrix}$$

this yields $\mathrm{Isom}^0(\widetilde{\mathcal{M}}, \widetilde{g}) \cong \mathcal{S} \ltimes_\pi \mathrm{He}(n)$. Finally, note that the first set of Killing vector fields treated as elements of $\mathfrak{he}(n)$ are precisely those vector fields with coefficients $\xi_{i,k} \in C^\infty(\mathbb{R})$ s.t. $\xi_i = (\xi_{i,1}, \ldots, \xi_{i,n}) \in \mathscr{E}$. As solutions $\xi \in \mathscr{E}$ and the additional flows Φ_τ as above are all defined on the whole real line this in particular shows that $\mathfrak{kill}(\widetilde{\mathcal{M}}, \widetilde{g}) \cong \mathfrak{kill}_c(\widetilde{\mathcal{M}}, \widetilde{g})$ with $\mathfrak{kill}_c(\widetilde{\mathcal{M}}, \widetilde{g})$ denoting the Lie subalgebra consisting of complete Killing vector fields. As $\mathfrak{kill}_c$ is the Lie algebra corresponding to $\mathrm{Isom}^0(\widetilde{\mathcal{M}}, \widetilde{g})$ this completes the proof. □

Let us give an explicit example with $n = 3$ for which we can explicitly compute the Lie group $\mathcal{S}$ occurring in Theorem 2.33. As it turns out, this example is a special case in $n = 3$ for the subsequent corollary on the maximal dimension of $\mathrm{Isom}^0(\widetilde{\mathcal{M}}^{(n+2)}, \widetilde{g})$.

Example 2.34. *Let $A \in \mathrm{End}(V)$, $\dim V = 3$, be given through*

$$A = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{pmatrix}$$

for $\lambda, \mu \in \mathbb{R}^*$ *with* $2\lambda + \mu = 0$*. Then, up to multiplication by scalars,*

$$F = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is the only non-trivial possible solution to (2.35)*. Using Rodrigues's formula for the matrix exponential* $\exp : \mathfrak{so}(3) \longrightarrow \mathrm{SO}(3)$*, given through*

$$\exp(X) := \exp \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} = \mathbb{I} + \frac{\sin\theta}{\theta} X + \frac{(1-\cos\theta)}{\theta^2} X^2$$

where $\theta := \sqrt{a^2 + b^2 + c^2}$*, we conclude* $\ker(t \mapsto \exp(tF)) = 2\pi\mathbb{Z}$*. Hence* $\{\Phi_t\} \cong \mathbb{R}/2\pi\mathbb{Z} = \mathbb{S}^1$ *and thus* $\mathrm{Isom}^0(\widetilde{\mathcal{M}}^{(n+2)}, \widetilde{g}) = \mathbb{S}^1 \ltimes \mathrm{He}(3)$*.*

Indeed, if we generalize the preceding example, we may state the following corollary from Theorem 2.33.

Corollary 2.35. *Let* $\dim V = n$ *and* $A \in \mathrm{End}(V)$ *be a traceless self-adjoint operator such that* $V = E_1 \oplus E_2$ *is a splitting of* V*, where* E_1*,* E_2 *are the eigenspaces of* A *such that* $\dim E_1 = n-1$*,* $\dim E_2 = 1$*. Then* $\mathrm{Isom}^0(\widetilde{\mathcal{M}}^{(n+2)}, \widetilde{g})$ *is a Lie subgroup of the full isometry group of dimension* $\frac{(n+2)(n+3)}{2} - \dim \mathfrak{he}(n)$*.*

Proof. The only restriction in the choice of an $F \in \mathfrak{so}(n)$ satisfying (2.35) is given through the distinct eigenvalues for the two eigenspaces of A, i.e. $F : V = E_1 \oplus E_2 \longrightarrow E_1 \oplus E_2$ has to leave the eigenspaces invariant. As F is skew-adjoint and $\dim E_1 = 1$ we obtain for the Lie algebra $\mathfrak{s} := \mathrm{span}\{F \in \mathfrak{so}(n) \mid [A, F] = 0\}$ that $\dim \mathfrak{s} = \dim \mathfrak{so}(n-1) = \frac{(n-1)(n-2)}{2}$. Hence

$$\begin{aligned} \dim \mathfrak{isom}(\widetilde{\mathcal{M}}^{(n+2)}, \widetilde{g}) &= \dim \mathfrak{he}(n) + \dim \mathfrak{s} \\ &= (2n+1) + \tfrac{(n-1)(n-2)}{2} = \frac{(n+2)(n+3)}{2} - (2n+1). \end{aligned}$$

This completes the proof. □

Remark 2.36. *We point out that self-adjoint operators* $A \in \mathrm{End}(V)$ *whose two eigenspaces yield a decomposition* $V = E_1 \oplus E_2$ *with* $\dim E_2 = 1$ *do not occur in the explicit constructions in [DR10]. This is since they could not be shown to yield compact quotients in dimensions* $3j + 2$ *with* $j > 1$*.*

More precisely, the examples in [DR10] contain operators $A \in \mathrm{End}(V)$ *with at most three distinct eigenvalues* $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ *with* $j = \dim E_1 = \dim E_2 = \dim E_3 = \frac{n}{3}$*. Hence, choosing* $\lambda_1 = \lambda_2 \neq \lambda_3$*, we can construct by Corollary 2.35 complete, compact Lorentzian manifolds with essentially parallel Weyl tensor whose full isometry group has dimension at least* $\frac{(n+2)(n+3)}{2} - (2 \cdot (\frac{n}{3})^2 + n + 2)$*.*

Proof. Choosing $\lambda_1 = \lambda_2 \neq \lambda_3$, the required $F \in \mathfrak{so}(n)$ has to commute with the operator A, whence $F(E_1 \oplus E_2) = E_1 \oplus E_2$ and $F(E_3) = E_3$. Thus it holds

$$\dim \mathfrak{isom}(\widetilde{\mathcal{M}}^{(n+2)}, \widetilde{g}) = \dim \mathfrak{he}(n) + \dim \mathfrak{s}$$

$$= \quad (2n+1) + \dim \mathfrak{so}(j) + \mathfrak{so}(2j) = (6j+1) + \tfrac{5j^2-3j}{2} = \tfrac{5j^2+9j+2}{2}.$$

Since $\frac{(n+2)(n+3)}{2} - \frac{5j^2+9j+2}{2} = 2 \cdot (\frac{n}{3})^2 + n + 2$ this completes the proof. $\square$

Remark 2.37. *We remark that Theorem 2.33 answers a question of* Derdziński *posed in [DR07] in the Lorentzian case. Namely it shows that a compact Lorentzian manifold with essentially parallel Weyl tensor cannot be locally homogeneous.*

3 LORENTZIAN MANIFOLDS WITH HIGHEST FIRST BETTI NUMBER

Within this chapter we consider orientable decent Lorentzian manifolds. We aim to prove a Lorentzian version of the *classical Bochner result* by which any compact, oriented *Riemannian* manifold $\mathcal{N}$ with non-negative Ricci curvature has first Betti number[1] $b_1(\mathcal{N}) \leq \dim \mathcal{N}$ and $b_1(\mathcal{N}) = \dim \mathcal{N}$ if and only if it is isometric to the flat torus [Peto6, Ch. 7, Corollary 19]. Our results where motivated by the following theorem.

Theorem 3.1 ([Lär11, Theorem 2.82]). *Let $(\mathcal{M}, g)$ be an orientable decent Lorentzian manifold such that the leaves of $\mathbb{L}^\perp$ are compact[2] and the Ricci curvature of g is non-negative on $\mathbb{L}^\perp \times \mathbb{L}^\perp$. Then the Betti numbers are restricted to*

$$\mathrm{b} \leq b_1(\mathcal{M}) \leq \dim(\mathcal{M}) - 1 + \mathrm{b},$$

where $\mathrm{b} \in \{0, 1\}$ and $\mathrm{b} = 1$ if $\mathcal{M}$ is compact or $\mathrm{b} = 0$ if $\mathcal{M}$ is non-compact.

The main intention of this chapter is to prove that the upper bounds in this theorem are taken if and only if the manifold $\mathcal{M}$ is diffeomorphic to a finite cover of the torus (in the compact case) or the product of the real line with the torus (in the non-compact case) and g has *light-like hypersurface curvature*, see Theorem 3.8.

Definition 3.2 ([Leio6]). *A decent Lorentzian manifold $(\mathcal{M}, g)$ is said to have* **light-like hypersurface curvature***, if and only if the curvature R^g satisfies*

$$R^g(X, Y)W \in \Gamma(\mathbb{L}) \tag{3.1}$$

for all $X, Y, W \in \Gamma(\mathbb{L}^\perp)$.

As one easily observes, Lorentzian manifolds with light-like hypersurface curvature are generalizations of pp- resp. pr-waves by weakening the curvature condition imposed. Although with the weaker condition (3.1) the curvature is still very degenerate, the following fact concerning the holonomy holds true.

Remark 3.3 ([Leio6, Proposition 12]). *For any of the four types of indecomposable, non-irreducible Lorentzian holonomy (see Section 1.1) and any Riemannian holonomy algebra $\mathfrak{g}$ there is a Lorentzian manifold $(\mathcal{M}, g)$ with light-like hypersurface curvature such that the holonomy of $(\mathcal{M}, g)$ is of the given type and its screen holonomy is equal to $\mathfrak{g}$.*

In order to prove the mentioned Bochner result we need some preliminary observations. Let $(\mathcal{M}^{(n+2)}, g)$ be an $(n+2)$-dimensional orientable decent Lorentzian manifold with recurrent light-like vector field $V \in \Gamma(\mathbb{L})$. Recall from Section 1.3 that by fixing a

1 We define the first Betti number of any manifold $\mathcal{M}$ to be the rank of $H^1(\mathcal{M}, \mathbb{R})$.

2 We remind the reader that for decent Lorentzian manifolds, the leaves are either closed or dense due to Theorem 1.25.

realization $\mathbb{S}$ of the screen bundle and denoting by g^R the associated Riemannian metric on $\mathcal{M}$ (see (1.19) on page 20), we obtain a transversally oriented Riemannian flow $\mathscr{F}$ on the oriented manifold $\mathcal{L}^\perp$ by the flow of V and the Riemannian metric h obtained by restricting g^R to $\mathcal{L}^\perp$, cf. Lemma 1.32 and Proposition 1.31. *Throughout this chapter we will denote this Riemannian flow by* $(\mathcal{L}^\perp, \mathscr{F}, h)$. As a first observation we get:

Lemma 3.4. *Let $\mathcal{L}^\perp$ be a compact $(n+1)$-dimensional leaf of $\mathbb{L}^\perp$. Then*

$$H^1_{\mathrm{dR}}(\mathcal{L}^\perp) = H^1_B(\mathscr{F}) \oplus H$$

for H a subgroup of $H^n_B(\mathscr{F}) \in \{0, \mathbb{R}\}$.

Proof. Since $(\mathcal{L}^\perp, \mathscr{F}, h)$ is a Riemannian flow, by Theorem 1.38, there exists a bundle-like metric $\widehat{h}$ such that the mean-curvature 1-form κ to $(\mathcal{L}^\perp, \mathscr{F}, \widehat{h})$ is basic-harmonic. By the Gysin sequence for $(\mathcal{L}^\perp, \widehat{h})$, cf. Theorem 1.42, we obtain an exact sequence

$$0 \longrightarrow H^1_B(\mathscr{F}) \longrightarrow H^1_{\mathrm{dR}}(\mathcal{L}^\perp) \longrightarrow H^0_\kappa(\mathscr{F}) \longrightarrow H^2_B(\mathscr{F}) \longrightarrow \dots \tag{3.2}$$

and by taking into account that $H^0_\kappa(\mathscr{F}) \cong H^n_B(\mathscr{F})$, by the Poincaré duality Theorem 1.41 this translates into the long exact sequence

$$0 \longrightarrow H^1_B(\mathscr{F}) \longrightarrow H^1_{\mathrm{dR}}(\mathcal{L}^\perp) \xrightarrow{\Phi} H^n_B(\mathscr{F}) \longrightarrow H^2_B(\mathscr{F}) \longrightarrow \dots \tag{3.3}$$

and thus we obtain the short exact sequence

$$0 \longrightarrow H^1_B(\mathscr{F}) \longrightarrow H^1_{\mathrm{dR}}(\mathcal{L}^\perp) \longrightarrow H \longrightarrow 0 \tag{3.4}$$

for $H := \operatorname{im}\Phi \subset H^n_B(\mathscr{F})$. In particular this is a short exact sequence of vector spaces and hence splits as a direct sum. Since $H^n_B(\mathscr{F}) \in \{0, \mathbb{R}\}$ by Theorem 1.39 this completes the proof. □

By assuming non-negativity of the Ricci curvature on $\mathbb{L}^\perp \times \mathbb{L}^\perp$, we obtain the following estimation for the dimensions of $H^1_{\mathrm{dR}}(\mathcal{L}^\perp)$ and $H^1_B(\mathscr{F})$. This is precisely [Lär11, Proposition 2.81]. However, to be self-contained, make the upcoming proofs more precise and to fix notation we will present its full proof here.

Lemma 3.5. *Let $\mathcal{L}^\perp$ be a compact $(n+1)$-dimensional leaf of $\mathbb{L}^\perp$ and $\mathrm{Ric}^g|_{\mathbb{L}^\perp \times \mathbb{L}^\perp} \geq 0$. Then $b_1(\mathcal{L}^\perp) \leq \dim H^1_B(\mathscr{F}) + 1 \leq \dim \mathcal{L}^\perp$.*

Proof. Again, consider the Riemannian flow $(\mathcal{L}^\perp, \mathscr{F}, h)$ and the bundle-like metric $\widehat{h}$ such that the mean-curvature 1-form κ of $(\mathcal{L}^\perp, \mathscr{F}, \widehat{h})$ is basic-harmonic, constituted by Theorem 1.38. In particular, the induced metrics on $\Sigma|_{\mathcal{L}^\perp} = \mathcal{L}^\perp / T\mathscr{F}$ coincide and hence so do the induced transversal connections. Defining by $\widehat{\mathbb{S}} := \ker \widehat{h}(V, \cdot)$ a realization of $\Sigma|_{\mathcal{L}^\perp}$, the transversal Levi-Civita connection $\nabla^T : \Gamma(\widehat{\mathbb{S}}) \longrightarrow \Gamma(T^*\mathcal{L}^\perp \otimes \widehat{\mathbb{S}})$ is given by

$$\nabla^T_X Y := \begin{cases} \mathrm{pr}_{\widehat{\mathbb{S}}}(\widehat{\nabla}_X Y), & X \in \Gamma(\widehat{\mathbb{S}}), \\ \mathrm{pr}_{\widehat{\mathbb{S}}}([X, Y]), & X \in \Gamma(\mathbb{L}), \end{cases} \tag{3.5}$$

for any $Y \in \Gamma(\widehat{\mathbb{S}})$, where $\widehat{\nabla}$ is the Levi-Civita connection to $\widehat{h}$ (see (1.30) on page 27). Since $R^g(\cdot,\cdot)V \in \Gamma(\mathbb{L})$ and $g_{\widehat{\mathbb{S}}\times\widehat{\mathbb{S}}} = h|_{\widehat{\mathbb{S}}\times\widehat{\mathbb{S}}}$ we have that

$$\mathrm{Ric}^T_{\widehat{\mathbb{S}}\times\widehat{\mathbb{S}}} = \mathrm{Ric}^g|_{\widehat{\mathbb{S}}\times\widehat{\mathbb{S}}} \geq 0. \tag{3.6}$$

Because any class in $H^1_B(\mathscr{F})$ can be represented by a Δ_B-harmonic one-form $\alpha \in \Omega^1_B(L)$ due to Theorem 1.37, we choose such a basic-harmonic α for each generator of $H^1_B(\mathscr{F})$. To apply a Bochner argument we need an appropriate Weizenböck formula which is given by [HR10, Proposition 6.7] and by integration reads

$$0 = \int_{\mathcal{L}^\perp} ||\nabla^T\alpha||^2 + \int_{\mathcal{L}^\perp} \mathrm{Ric}^T(\alpha^\sharp, \alpha^\sharp),$$

where $\alpha^\sharp$ is the dual to α w.r.t. $\widehat{h}$. Hence $\nabla^T_X\alpha = 0$ for all $X \in \Gamma(T\mathcal{L}^\perp)$. This proves $\dim H^1_B(\mathscr{F}) \leq n = \dim \widehat{\mathbb{S}}$ and thus the second asserted inequality. For the first inequality we apply Lemma 3.4; this completes the proof. □

Applying these results we see that maximality of the dimension of the first Betti number of $\mathcal{L}^\perp$ and $\mathrm{Ric}^g|_{\mathbb{L}^\perp\times\mathbb{L}^\perp} \geq 0$ already yield that $\mathcal{L}^\perp$ is the torus with the curvature of $\nabla^g|_{\mathcal{L}^\perp}$ on $\mathcal{L}^\perp$ being light-like.

Proposition 3.6. *Let $(\mathcal{M}^{(n+2)}, g)$ be an oriented $(n+2)$-dimensional decent Lorentzian manifold such that $\mathrm{Ric}^g|_{\mathbb{L}^\perp\times\mathbb{L}^\perp} \geq 0$ and the leaves $\mathcal{L}^\perp$ of the codimension one foliation induced by $\mathbb{L}^\perp$ are compact. Then, $b_1(\mathcal{L}^\perp) \leq n+1$ and $b_1(\mathcal{L}^\perp) = n+1$ if and only if $\nabla^g|_{\mathcal{L}^\perp}$ has light-like curvature and $\mathcal{L}^\perp$ is diffeomorphic to the torus.*

Proof. The proof works as follows: Let $b_1(\mathcal{L}^\perp) = n+1$. We will define for every realization of the screen bundle another screen distribution *on* $\mathcal{L}^\perp$ which is horizontal and involutive. The associated Riemannian metric h to this screen distribution then has non-negative Ricci curvature by Proposition 1.21 and hence $(\mathcal{L}^\perp, h)$ is the flat torus implying $\nabla^g|_{\mathcal{L}^\perp}$ to have light-like curvature by (1.27).

Step 1. Fix a realization of the screen bundle and hence a Riemannian flow $(\mathcal{L}^\perp, \mathscr{F}, h)$ on $\mathcal{L}^\perp$. Recall that $H^n_B(\mathscr{F}) \in \{0, \mathbb{R}\}$ and assume $H^n_B(\mathscr{F}) = 0$. Then we would have Lemma 3.4 that $H^1_B(\mathscr{F}) \cong H^1_{\mathrm{dR}}(\mathcal{L}^\perp)$ which is impossible due to $b_1(\mathcal{L}^\perp) = n+1$ and Lemma 3.5. Hence $H^n_B(\mathscr{F}) = \mathbb{R}$ and Theorem 1.39 implies that there is a metric $\widehat{h}$ with $\widehat{h}(V,V) = 1$ turning $(\mathcal{L}^\perp, \mathscr{F}, \widehat{h})$ into an *isometric* Riemannian flow, i.e. V is a $\widehat{h}$-Killing field and therefore $\mathscr{L}_V\chi = 0$ and $\kappa = 0$ for $\chi := \widehat{h}(V,\cdot)$, cf. Remark 1.30.

Step 2. By Lemma 3.5 and $b_1(\mathcal{L}^\perp) = n+1$ we infer $\dim H^1_B(\mathscr{F}) = n$. For $(\mathcal{L}^\perp, \mathscr{F}, \widehat{h})$ being an *isometric* Riemannian flow, the Gysin long exact sequence (3.3) is given as

$$0 \longrightarrow H^1_B(\mathscr{F}) \longrightarrow H^1_{\mathrm{dR}}(\mathcal{L}^\perp) \xrightarrow{\iota_*} H^0_B(\mathscr{F}) \xrightarrow{\delta} H^2_B(\mathscr{F}) \longrightarrow \ldots \tag{3.7}$$

where $\iota = (V \lrcorner \cdot)$ and $\delta = [d\chi \wedge \cdot]$, cf. Theorem 1.43. Note that since $H^0_\kappa(\mathscr{F}) \cong H^n_B(\mathscr{F})$ by the Poincaré duality Theorem 1.41 and $\kappa = 0$ we have

$$H^0_B(\mathscr{F}) = H^0_\kappa(\mathscr{F}) \cong H^n_B(\mathscr{F}) = \mathbb{R}.$$

This together with (3.7) implies the short exact sequence

$$0 \longrightarrow H^1_B(\mathscr{F}) \longrightarrow H^1_{\mathrm{dR}}(\mathcal{L}^\perp) \longrightarrow \mathrm{im}\,\iota_* \longrightarrow 0 \tag{3.8}$$

with $\operatorname{im}\iota_* = \mathbb{R}$ since $b_1(\mathcal{L}^\perp) = n + \dim\operatorname{im}\iota_*$. Hence, by the exactness of (3.7), $\ker\delta = \mathbb{R}$ and so $0 = \delta([1]) = [1 \cdot d\chi]$, i.e. $[d\chi]$ vanishes in $H^2_B(\mathscr{F})$. In this case, $d\chi = d_B\alpha$ for some $\alpha \in \Omega^1_B(\mathcal{L}^\perp)$. Then, $\omega := \chi - \alpha \in \Omega^1(\mathcal{L}^\perp)$ is closed w.r.t. d and $\omega(V) = 1$. We define

$$\mathbb{S} := \ker\omega. \tag{3.9}$$

Obviously, $\mathbb{S}$ is involutive and it is horizontal since $\mathscr{L}_V\omega = \mathscr{L}_V\chi - \mathscr{L}_V\alpha = 0$ as α is a basic 1-form and $\mathscr{L}_V\chi = 0$ by Step 1.

Step 3. Define a Riemannian metric Θ on $\mathcal{L}^\perp$ associated to g and $\mathbb{S}$ by

$$\Theta(X,Y) := \begin{cases} 1, & X = Y = V \\ g(X,Y), & X,Y \in \Gamma(\mathbb{S}) \\ 0, & X \in \Gamma(\mathbb{L}) \text{ and } Y \in \Gamma(\mathbb{S}) \text{ or } Y \in \Gamma(\mathbb{L}) \text{ and } X \in \Gamma(\mathbb{S}) \end{cases}$$

and linear extension. Since $\mathbb{S}$ is horizontal and involutive, Proposition 1.21 yields

$$\mathrm{Ric}^\Theta = \mathrm{Ric}^g|_{\mathbb{L}^\perp \times \mathbb{L}^\perp} \geq 0.$$

Therefore, $(\mathcal{L}^\perp, \Theta)$ turns into a compact, orientable Riemannian manifold with non-negative Ricci-curvature and is thus isometric to the flat torus. In particular, equation (1.27) in Proposition 1.21 implies that $(\mathcal{L}^\perp, \nabla^g|_{\mathcal{L}^\perp})$ has light-like curvature. $\square$

The last lemma we need relates the dimension of $H^1_{\mathrm{dR}}(\mathcal{M})$ with the dimension of $H^1_{\mathrm{dR}}(\mathcal{L}^\perp)$ for a compact leaf of the codimension one foliation $\mathbb{L}^\perp$.

Lemma 3.7. *Let $(\mathcal{M}^{(n+2)}, g)$ be an $(n+2)$-dimensional decent Lorentzian manifold such that the leaves $\mathcal{L}^\perp$ of the codimension one foliation induced by $\mathbb{L}^\perp$ are compact. Then,*

(i) $b_1(\mathcal{M}) \leq b_1(\mathcal{L}^\perp) + 1$.

(ii) $\pi_1(\mathcal{M}) \cong \mathbb{Z}\,{}_\varphi\!\ltimes \pi_1(\mathcal{L}^\perp)$ *for a homomorphism* $\varphi \in \mathrm{Hom}(\mathbb{Z}, \mathrm{Aut}(\pi_1(\mathcal{L}^\perp)))$ *and hence*

$$H_1(\mathcal{M}, \mathbb{Z}) \cong \mathbb{Z} \oplus H_1(\mathcal{L}^\perp, \mathbb{Z})/K,$$

for the subgroup K of $\pi_1(\mathcal{L}^\perp)$ generated by the elements $\varphi(k)([\gamma]) \cdot [\gamma]^{-1}$ with $k \in \mathbb{Z}$ and $[\gamma] \in \pi_1(\mathcal{L}^\perp)$.

Proof. Since $\mathbb{L}^\perp$ is the kernel of σ for $\sigma := g(V, \cdot)$ and $d\sigma = 0$, all leaves have trivial leaf holonomy by Theorem 1.24 (i). Moreover, by Theorem 1.24 (iii), the compactness assumption for the leaves implies that $\mathcal{M}$ fibers over $\mathbb{S}^1$ or $\mathbb{R}$ with each leaf being given as a fiber of the fibration. Hence, if $\mathcal{M}$ is not compact, $\mathcal{M}$ fibers over $\mathbb{R}$ and thus $\mathcal{M} \simeq \mathbb{R} \times \mathcal{L}^\perp$ for a fixed leaf. Otherwise, if $\mathcal{M}$ is compact, $\mathcal{M}$ fibers over $\mathbb{S}^1$ and the long exact sequence of homotopy groups for fibrations [Hat02, Theorem 4.41] yields

$$0 \longrightarrow \pi_1(\mathcal{L}^\perp) \longrightarrow \pi_1(\mathcal{M}) \longrightarrow \mathbb{Z} \longrightarrow 0$$

and hence $\pi_1(\mathcal{M}) \cong \mathbb{Z}\,{}_\varphi\!\ltimes \pi_1(\mathcal{L}^\perp)$ for some $\varphi \in \mathrm{Hom}(\mathbb{Z}, \mathrm{Aut}(\pi_1(\mathcal{L}^\perp)))$ since $\mathbb{Z}$ is free. By the Hurewicz theorem we have $H_1(\mathcal{M}, \mathbb{Z}) \cong \pi_1(\mathcal{M})_{\mathrm{ab}}$, where for any group G we denote with $G_{\mathrm{ab}} := G/[G,G]$ its Abelization. However, for any semi-direct product $G = A\,{}_\psi\!\ltimes B$ for some $\psi \in \mathrm{Hom}(A, \mathrm{Aut}(B))$ there is the identity

$$G_{\mathrm{ab}} = A_{\mathrm{ab}} \oplus B_{\mathrm{ab}}/H$$

with H denoting the subgroup of B generated by elements $\psi(a)b \cdot b^{-1}$ with $a \in A$ and $b \in B$, cf. [GG09, Proposition 3.3]. Therefore, we see that

$$H_1(\mathcal{M}, \mathbb{Z}) \cong \pi_1(\mathcal{M})_{\mathrm{ab}} \cong \mathbb{Z} \oplus H_1(\mathcal{L}^\perp, \mathbb{Z})/K$$

for the subgroup K generated by the elements $\varphi(k)([\gamma]) \cdot [\gamma]^{-1}$, $k \in \mathbb{Z}$, $[\gamma] \in \pi_1(\mathcal{L}^\perp)$. In particular, $b_1(\mathcal{M}) = 1 + \operatorname{rank} H_1(\mathcal{L}^\perp, \mathbb{Z}) - \operatorname{rank} K \leq b_1(\mathcal{L}^\perp) + 1$. □

As a subsumption of the preceding results we obtain the announced Bochner-type theorem for decent Lorentzian manifolds.

Theorem 3.8. *Let $(\mathcal{M}^{(n+2)}, g)$ be an orientable $(n+2)$-dimensional decent Lorentzian manifold. Assume that the leaves of the codimension one foliation induced by the distribution $\mathbb{L}^\perp$ are compact and $\operatorname{Ric}|_{\mathbb{L}^\perp \times \mathbb{L}^\perp} \geq 0$.*

(i) If $\mathcal{M}$ is compact then $b_1(\mathcal{M}) \leq n+2$ and $b_1(\mathcal{M}) = n+2$ if and only if $\mathcal{M}$ is – up to finite cover – diffeomorphic (homeomorphic[3] if $\dim \mathcal{M} = 4$) to the torus and g has light-like hypersurface curvature.

(ii) If $\mathcal{M}$ is non-compact then $b_1(\mathcal{M}) \leq n+1$ and $b_1(\mathcal{M}) = n+1$ if and only if $\mathcal{M}$ is isometric to $\mathbb{R} \times \mathbb{T}^{n+1}$ and g has light-like hypersurface curvature.

In both cases, the leaves of $\mathbb{L}^\perp$ are all diffeomorphic to the torus $\mathbb{T}^{n+1}$.

Proof. *Ad* (i). Since $b_1(\mathcal{M}) = n+2$ we infer $b_1(\mathcal{L}^\perp) = n+1$ by Lemma 3.7 (i) taking into account that $b_1(\mathcal{L}^\perp) \leq \dim \mathcal{L}^\perp = n+1$ by Lemma 3.5. Hence, all assumptions of Proposition 3.6 are satisfied. We obtain, that the connection $\nabla^g|_{\mathcal{L}^\perp}$ on each leaf $\mathcal{L}^\perp$ induced by the Levi-Civita connection of g has light-like curvature. By [Lei06, Proposition 6] this is equivalent for $(\mathcal{M}, g)$ to have light-like hypersurface curvature.

Since $\mathcal{L}^\perp = \mathbb{T}^{n+1}$ and hence $\mathcal{M}$ fibers over $\mathbb{S}^1$ with toric fibers, the long exact sequence of homotopy groups for fibrations implies that for $k > 1$ all homotopy groups $\pi_k(\mathcal{M})$ vanish. Therefore, $\mathcal{M}$ is a $K(\pi_1(\mathcal{M}), 1)$-space (see page 37), while

$$\pi_1(\mathcal{M}) \cong \mathbb{Z} {}_\varphi\!\ltimes \pi_1(\mathcal{L}^\perp) \cong \mathbb{Z} {}_\varphi\!\ltimes \mathbb{Z}^{n+1}$$

by Lemma 3.7 (ii) and since $\pi_1(\mathcal{L}^\perp) = \mathbb{Z}^{n+1}$. By assumption, $H_1(\mathcal{M}, \mathbb{Z}) = \mathbb{Z}^{n+2} \oplus \mathrm{Tor}$ and hence

$$\mathbb{Z}^{n+2} \oplus \mathrm{Tor} = H_1(\mathcal{M}, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}^{n+1}/K$$

by Lemma 3.7 (ii). But this equation can only hold, if K is trivial. Namely, comparing the ranks of the left and the right hand side, we observe

$$n + 2 = 1 + (n+1) - \operatorname{rank} K \Longleftrightarrow \operatorname{rank} K = 0.$$

As K is a subgroup of $\pi_1(\mathcal{L}^\perp) \cong \mathbb{Z}^{n+1}$ it has no torsion and hence must be trivial. Therefore, $\mathcal{M}^{n+2}$ is a $K(\mathbb{Z}^{n+2}, 1)$-space and is thus homotopy-equivalent to the torus [Hat02, Theorem 1B.8]. But in the case of the torus, for $\dim \mathcal{M} \leq 3$ this is even equivalent for

3 In dimension four, $\mathcal{M}$ is only known to be *homeomorphic* to the torus and to our best knowledge it seems to be an open problem in geometric topology if $\mathcal{M}$ must be also diffeomorphic to the torus.

$\mathcal{M}$ itself[4] or some finite cover (if $\dim \mathcal{M} > 4$) to be diffeomorphic to the standard torus [WR99, Page 236]. For $\dim \mathcal{M} = 4$ we can only conclude that $\mathcal{M}$ is homeomorphic to $\mathbb{T}^4$ [FQ90, Chapter 11.5].

Ad (ii). The second part is straightforward since $\mathcal{M} \simeq \mathbb{R} \times \mathcal{L}^\perp$ for a fixed leaf $\mathcal{L}^\perp$ by Theorem 1.24 (iii) and hence $n+1 = b_1(\mathcal{M}) = b_1(\mathcal{L}^\perp)$ so Proposition 3.6 applies. □

Remark 3.9. *If $\mathcal{M}$ is non-orientable, the statements asserted in the theorem inherit to the 2-fold orientation covering $\widehat{\mathcal{M}}$ for $\mathcal{M}$ equipped with the metric $\widehat{g}$ being the pull-back of g.*

In particular we can still conclude $b_1(\mathcal{M}) \leq n+2$ since $b_1(\widehat{\mathcal{M}}) = b_1(\mathcal{M}) + b_{n+1}(\mathcal{M})$, cf. [Bra69]. Hence the conclusions of Theorem 3.8 concerning the upper bounds of the Betti numbers and the curvature hold for $(\mathcal{M}, g)$, too.

4 For the 3-dimensional case, WALDHAUSEN [Wal68] proved this for *Haken manifolds* and thus in particular for 3-dimensional closed manifolds fibering over the circle. Note that for 3-dimensional manifolds classifications up to diffeomorphism and homeomorphism coincide [Moi52].

4 TOTAL SPACES OF CIRCLE BUNDLES WITH SPECIAL HOLONOMY

4.1 CONSTRUCTION

We will construct Lorentzian manifolds with special holonomy as total spaces of principal bundles over the circle. In [Lär11], this construction was already used by the same motivation to produce examples with non-trivial topology (i.e. manifolds not diffeomorphic to a product of $\mathcal{L}_1 \times \mathcal{L}_2 \times \mathcal{B}$ with $\mathcal{L}_i \in \{\mathbb{R}, \mathbb{S}^1\}$) and full holonomy of type 1 or 2. We will provide different examples here and obtain in particular complete Ricci-flat Lorentzian manifolds with prescribed full holonomy. Moreover, our constructions will provide *complete* Lorentzian manifolds with holonomy of type 4 which to our best knowledge are the first of this kind.

Let $(\mathcal{N}^{(n+1)}, h)$ be an $(n+1)$-dimensional *Riemannian* manifold and $\omega \in H^2(\mathcal{N}, \mathbb{Z})$ an integral cohomology class. For the $\mathbb{S}^1$-bundle $\pi : \mathcal{M} \longrightarrow \mathcal{N}$ with first Chern class $c_1(\mathcal{M}) = \omega$ consider the following Lorentzian metric g on $\mathcal{M}$. Take any closed 2-form $\overline{\Psi} \in \Omega^2(\mathcal{N})$ s.t. $\overline{\Psi}$ represents ω in the de Rham cohomology[1] and a corresponding connection $A \in \Omega^1(\mathcal{M}, \mathrm{i}\mathbb{R})$ with curvature $F^A = dA = -2\pi \mathrm{i} \pi^* \overline{\Psi}$. Then, for any nowhere vanishing closed 1-form $\eta \in \Omega^1(\mathcal{N})$ and any function $f \in C^\infty(\mathcal{M})$ define

$$g := 2\mathrm{i}A \odot \pi^*\eta + f \cdot \pi^*\eta \odot \pi^*\eta + \pi^* h. \tag{4.1}$$

Then, $(\mathcal{M}^{(n+2)}, g)$ is an $(n+2)$-dimensional Lorentzian manifold.

Henceforth, we write $\Psi := \pi \cdot \overline{\Psi}$, and thus $F^A = -2\mathrm{i}\pi^*\Psi$. To refer to this construction we make the following definition.

Definition 4.1. *The Lorentzian manifold* $(\mathcal{M}^{(n+2)}, g)$ *with* g *chosen as in* (4.1) *is called* **manifold of type** (Ψ, A, η, f) **over** $(\mathcal{N}, h)$.

For the upcoming calculations we will use the following local frame on $(\mathcal{M}, g)$. Let $x = \pi(y) \in \mathcal{N}$ be an arbitrary point on $\mathcal{N}$. On $\mathcal{N}$ we have the global vector field $E_\eta := \frac{\eta^\sharp}{||\eta^\sharp||_h^2}$ and on $\mathcal{M}$ the fundamental vector field $\xi \in \Gamma(T\mathcal{M})$ corresponding to the $\mathbb{S}^1$-action, i.e.

$$\xi(z) := \widetilde{\mathrm{i}}(z) = \frac{d}{dt}(z \cdot \exp(t \cdot \mathrm{i}))|_{t=0},$$

$z \in \mathcal{M}$, which is light-like w.r.t. g. Locally around $x \in \mathcal{U} \subset \mathcal{N}$, we may choose a frame $E_1, \dots, E_n, E_\eta$ s.t. $h(E_i, E_j) = \delta_{ij}$ and $\ker \eta = \mathrm{span}\{E_1, \dots, E_n\} \perp_h \mathbb{R}E_\eta$. Taking its horizontal lifts $E_i^* \in \Gamma(T\mathcal{M}|_{\pi^{-1}(\mathcal{U})})$ we thus obtain a local orthonormal frame on $(\mathcal{M}, g)$:

$$\mathrm{e}_i := E_i^*, \quad \mathrm{e}_+ := \zeta + \tfrac{1}{2}H\xi, \quad \mathrm{e}_- := \mathrm{e}_+ + \xi, \tag{4.2}$$

1 Throughout this chapter we will write $\omega \in H^*_{\mathrm{dR}}(\mathcal{N}) \cap H^*(\mathcal{N}, \mathbb{Z})$ to indicate that ω is chosen to be a class in the de Rham cohomology whose image under the isomorphism $H^*_{\mathrm{dR}}(\mathcal{N}) \cong H^*(\mathcal{N}, \mathbb{R})$ is actually in $H^*(\mathcal{N}, \mathbb{Z})$, i.e. the integral of ω over all singular cycles is an integer.

with $\zeta := E_\eta^*$, $H := (f + \frac{1}{||\eta^\sharp||_h^2} - 1)$ and $i = 1, \dots, n$. Then, $g(\mathrm{e}_i, \mathrm{e}_j) = \delta_{ij}$, $g(\mathrm{e}_i, \mathrm{e}_+) = g(\mathrm{e}_i, \mathrm{e}_-) = 0$, $g(\mathrm{e}_+, \mathrm{e}_+) = 1$ and $g(\mathrm{e}_-, \mathrm{e}_-) = -1$.

We do now proceed to calculate the Levi-Civita connection corresponding to g. Note that in all forthcoming formulas, the Latin indices i, j, k and ℓ run from 1 to n and ξ, $+$ denoted *as index within tensors* means plugging in the vector field ξ or e_+, respectively. Moreover, we omit the components with at least one e_--vector since these are immediate by the multi-linearity and Leibniz-rules of the objects in question.

Lemma 4.2. *Let $(\mathcal{M}^{(n+2)}, g)$ be of type (Ψ, A, η, f) over $(\mathcal{N}, h)$. Then,*

(i) $\nabla^g_{\mathrm{e}_i}\mathrm{e}_j = \overline{\nabla^h_{E_i} E_j}^* + (\frac{\mathrm{i}}{2} F^A(\mathrm{e}_i, \mathrm{e}_j) - h(E_\eta, \nabla^h_{E_i} E_j))\xi$,

(ii) $\nabla^g_{\mathrm{e}_+}\mathrm{e}_j = \overline{\nabla^h_{E_\eta} E_j}^* + \overline{\psi(E_j)}^* - (\mathrm{i}F^A(\mathrm{e}_j, \mathrm{e}_+) + \frac{1}{2} dH(\mathrm{e}_j))\xi$,

(iii) $\nabla^g_{\mathrm{e}_i}\mathrm{e}_+ = \overline{\nabla^h_{E_i} E_\eta}^* + \overline{\psi(E_i)}^*$,

(iv) $\nabla^g_{\mathrm{e}_+}\mathrm{e}_+ = \overline{\nabla^h_{E_\eta} E_\eta}^* + 2\overline{\psi(E_\eta)}^* - \frac{1}{2}\operatorname{grad}_g f - \frac{1}{2}\mathrm{e}_+(f)\xi$,

(v) $\nabla^g \xi = -\frac{1}{2}\xi(f) \cdot \pi^*\eta \otimes \xi$.

Here, $\psi \in \Omega^1(\mathcal{N}, T\mathcal{N})$ is defined as $h(\psi(E_i), E_j) := \Psi(E_i, E_j)$ and for any $X \in \Gamma(T\mathcal{N})$ we define $\overline{X} := \mathrm{pr}_{\ker \eta} X$. Hence, $X = \overline{X} + \eta(X) E_\eta$.

Proof. Since $\mathrm{e}_i = E_i^*$ are horizontal lifts, one has $[\mathrm{e}_i, \xi] = [\zeta, \xi] = 0$ and

$$\begin{aligned} [X^*, Y^*] &= [X, Y]^* - \widetilde{F^A(X, Y)} = [X, Y]^* + \mathrm{i}F^A(X^*, Y^*)\xi, \\ [\mathrm{e}_+, X^*] &= [\zeta, X^*] + \tfrac{1}{2}[H\xi, X^*] = [E_\eta, X]^* + (\mathrm{i}F^A(\zeta, X^*) - \tfrac{1}{2} dH(X^*))\xi, \end{aligned}$$

for all $X, Y \in \Gamma(T\mathcal{N})$. Moreover, by taking into account that $\eta \in \Omega^1(\mathcal{N})$ is closed and $\eta(E_\eta) \equiv 1$, we see that

$$\eta([E_i, E_j]) = \eta([E_\eta, E_i]) = 0$$

for $i = 1, \dots, n$. The formulas (i) to (v) are now immediate consequences of the Koszul formula for ∇^g. $\square$

In the case of the circle bundle metrics studied in this chapter we clearly have that $\mathbb{L} = \mathbb{R}\xi$ and, locally, $\mathbb{L}^\perp = \operatorname{span}\{\xi, \mathrm{e}_1, \dots, \mathrm{e}_n\}$. Moreover we have a canonical realization of the screen bundle. Namely, we may define by

$$Z := \frac{1}{2}\xi - \mathrm{e}_- \tag{4.3}$$

a light-like vector field with $g(\xi, Z) = 1$. Then, the metric g is non-degenerate on the plane $\operatorname{span}\{\xi, Z\}$ and we obtain a realization of the screen bundle by $\mathbb{S} := \operatorname{span}\{\xi, Z\}^{\perp_g}$ with nice properties:

Lemma 4.3. *Let $(\mathcal{M}^{(n+2)}, g)$ be of type (Ψ, A, η, f) over $(\mathcal{N}, h)$. Then, realizing the screen bundle as $\mathbb{S} = \{\xi, Z\}^{\perp_g}$, we obtain a horizontal realization of the screen bundle. Moreover, the screen distribution $\mathbb{S}$ is involutive if and only if $F^A|_{\ker \pi^*\eta \times \ker \pi^*\eta} = 0$ or, equivalently, $\eta \wedge \Psi = 0$.*

Proof. Of course, choosing, locally, the orthonormal frame (4.2), we clearly have that $\mathcal{S}|_{\pi^{-1}(\mathcal{U})} = \mathrm{span}\{e_1, \ldots, e_n\}$. Now, since $[\xi, e_i] = 0$, $\mathcal{S}$ is horizontal. Moreover, we have

$$[e_i, e_j] = [E_i, E_j]^* + \mathrm{i}F^A(e_i, e_j)\xi.$$

Consequently, $[e_i, e_j] \in \Gamma(\mathcal{S})$ if and only if $F^A(e_i, e_j) = 0$ or, equivalently, $\eta \wedge \Psi = 0$. □

Indeed, this does in general not imply that we cannot find another realization of the screen bundle which is involutive and horizontal. But in fact, one can construct manifolds of type (Ψ, A, η, f) over $(\mathcal{N}, h)$ for which such a realization cannot exist, cf. Example 4.8 or [Lär11, Corollary 2.71].

4.2 COMPLETENESS

We are now interested in conditions for which the Lorentzian manifolds $(\mathcal{M}^{(n+2)}, g)$ of type (Ψ, A, η, f) over $(\mathcal{N}, h)$ are complete. To establish criteria for completeness we preliminarily prove the following proposition which is a slight generalization of [RS94a, Proposition 2.1] in the Lorentzian case. However, for the sake of completeness, we present the proof here.

Proposition 4.4. *Let $(\mathcal{M}^{(n+2)}, g)$ be a Lorentzian manifold with timelike vector field X that satisfies the following three conditions:*

(i) $g(X, X)^{-1}$ is bounded on $\mathcal{M}$,

(ii) the Riemannian metric g^R given by

$$g^R|_{X^\perp \times X^\perp} = g,\ g^R(X, X) = -g(X, X),\ g^R|_{X^\perp \times X} = g^R|_{X \times X^\perp} = 0,$$

is complete.

Then for every inextensible g-geodesic $\gamma : [0, \varepsilon) \longrightarrow \mathcal{M}$, the map

$$t \in [0, \varepsilon) \longmapsto (\mathscr{L}_X g)(\dot\gamma(t), \dot\gamma(t))$$

is unbounded. (Here $\mathscr{L}_X g$ denotes the Lie-derivative of g along X.)

Proof. Let $\gamma : [0, \varepsilon) \longrightarrow \mathcal{M}$ be an inextensible g-geodesic with $0 < \varepsilon < \infty$. It suffices to show that the function $t \in [0, \varepsilon) \longmapsto g^R(\dot\gamma(t), \dot\gamma(t)) \in \mathbb{R}$ is bounded. Namely, in this case, $\{x_n := \gamma(t_n)\}$ for some $\{t_n\} \to \varepsilon$ is a d_R-Cauchy sequence, where d_R denotes the geodesic distance w.r.t. g^R. Since g^R is complete, the closure of $\{x_n\}$ is compact and so there exists a convergent subsequence to, say, $x \in \mathcal{M}$. But as $\{x_n\}$ is Cauchy, it converges to x, too, while the sequence $\{t_n\}$ with $t_n \to \varepsilon$ can be chosen arbitrarily. But then $\gamma : [0, \varepsilon) \longrightarrow \mathcal{M}$ is extensible beyond ε via $\lim_{t \to \varepsilon^-} \gamma(t) := x$ which is a contradiction.

Let $\widehat{X} := X/||X||$. Since $g(\widehat{X}, \widehat{X}) = -1$, $g^R(\widehat{X}, \widehat{X}) = 1$ and $\mathrm{pr}_{\widehat{X}^\perp} \dot\gamma = \dot\gamma + g(\widehat{X}, \dot\gamma)\widehat{X}$, we obtain

$$g(\dot\gamma, \dot\gamma) \;=\; g(\mathrm{pr}_{\widehat{X}^\perp} \dot\gamma - g(\widehat{X}, \dot\gamma)\widehat{X}, \mathrm{pr}_{\widehat{X}^\perp} \dot\gamma - g(\widehat{X}, \dot\gamma)\widehat{X})$$

$$\begin{aligned} &= g(\mathrm{pr}_{\widehat{X}^\perp}\dot\gamma, \mathrm{pr}_{\widehat{X}^\perp}\dot\gamma) - 2g(\mathrm{pr}_{\widehat{X}^\perp}\dot\gamma, g(\widehat{X},\dot\gamma)\widehat{X}) + g(\widehat{X},\dot\gamma)^2 g(\widehat{X},\widehat{X}) \\ &= g(\mathrm{pr}_{\widehat{X}^\perp}\dot\gamma, \mathrm{pr}_{\widehat{X}^\perp}\dot\gamma) - g(\widehat{X},\dot\gamma)^2 \end{aligned}$$

and

$$\begin{aligned} g^R(\dot\gamma,\dot\gamma) &= g^R(\mathrm{pr}_{\widehat{X}^\perp}\dot\gamma - g(\widehat{X},\dot\gamma)\widehat{X}, \mathrm{pr}_{\widehat{X}^\perp}\dot\gamma - g(\widehat{X},\dot\gamma)\widehat{X}) \\ &= g^R(\mathrm{pr}_{\widehat{X}^\perp}\dot\gamma, \mathrm{pr}_{\widehat{X}^\perp}\dot\gamma) - 2g^R(\mathrm{pr}_{\widehat{X}^\perp}\dot\gamma, g(\widehat{X},\dot\gamma)\widehat{X}) + g(\widehat{X},\dot\gamma)^2 g^R(\widehat{X},\widehat{X}) \\ &= g^R(\mathrm{pr}_{\widehat{X}^\perp}\dot\gamma, \mathrm{pr}_{\widehat{X}^\perp}\dot\gamma) + g(\widehat{X},\dot\gamma)^2. \end{aligned}$$

Since $g^R|_{X^\perp \times X^\perp} = g$ it follows

$$g^R(\dot\gamma,\dot\gamma) = g(\dot\gamma,\dot\gamma) + \frac{2}{g(X,X)} g(X,\dot\gamma)^2.$$

Since $g(\dot\gamma,\dot\gamma)$ is constant and $g(X,X)^{-1}$ is bounded, we are left to show that $g(X,\dot\gamma)$ is bounded on $[0,\varepsilon)$. We compute

$$\frac{d}{dt} g(X,\dot\gamma) = \frac{1}{2}(\mathscr{L}_X g)(\dot\gamma,\dot\gamma).$$

Hence, if $(\mathscr{L}_X g)(\dot\gamma,\dot\gamma)$ is bounded, so is $\frac{d}{dt} g(X,\dot\gamma)$ and consequently, also $g(X,\dot\gamma)$ on $[0,\varepsilon)$. $\square$

With the aid of the former proposition we can now prove the following.

Theorem 4.5. *Let $(\mathcal{M}^{(n+2)}, g)$ be of type (Ψ, A, η, f) over $(\mathcal{N}, h)$ with compact base $\mathcal{N}$ s.t. the function $f \in C^\infty(\mathcal{M})$ is constant along the fibers and $\eta^\sharp$ is a Killing field on $(\mathcal{N}, h)$. If, moreover, $\Psi(\eta^\sharp, \cdot) = 0$ then $(\mathcal{M}^{(n+2)}, g)$ is complete.*

In particular, on $\mathcal{M}$ there exists a nowhere-vanishing timelike Killing vector field if it additionally holds $\zeta(f) = 0$.

Proof. We define a vector field $K \in \Gamma(T\mathcal{M})$ by

$$K := \zeta + \tfrac{C}{2} \cdot \xi \quad \text{with constant} \quad C := \max_{\mathcal{M}} g(\zeta,\zeta) + \varepsilon \in \mathbb{R}, \tag{4.4}$$

where $\varepsilon > 0$ is arbitrarily chosen.

To apply Proposition 4.4, we have to show that $K \in \Gamma(T\mathcal{M})$ is timelike as the conditions (i) and (ii) are satisfied since $\mathcal{M}$ is compact.

For the length of K we get

$$g(K,K) < -\varepsilon < 0$$

due to the definition of $C \in \mathbb{R}$.

Since $\Psi(\eta^\sharp, \cdot) = 0$, we obtain by the formulas in Lemma 4.2, and the fact that

$$\nabla^g K = \nabla^g \zeta = \nabla^g \mathrm{e}_+ - \tfrac{1}{2} dH \otimes \xi$$

(since $\nabla^g \xi = 0$) that the Lie-derivative $\mathscr{L}_K g$ is given by

$$\mathscr{L}_K g = \frac{1}{2}\zeta(f)\pi^*\eta \odot \pi^*\eta.$$

Since $\pi^*\eta = -g(\xi, \cdot)$ is ∇^g-parallel and $\zeta(f)$ is bounded as $\mathcal{M}$ is compact, there is no inextensible geodesic on $(\mathcal{M}, g)$ by Proposition 4.4, hence completeness follows. $\square$

As we will see in the next section there are quite a lot of examples that fulfill the assumptions made in the previous theorem and are hence geodesically complete. However, the assumption $\Psi(\eta^\sharp, \cdot) = 0$ is not absolutely necessary. Indeed, the next proposition gives examples for compact manifolds of type (Ψ, A, η, f) over $(\mathcal{N}, h)$ with compact base $\mathcal{N}$ and $\Psi(\eta^\sharp, \cdot) \neq 0$ which are complete, too.

Proposition 4.6. *Let $(\mathcal{M}^{(n+2)}, g)$ be of type (Ψ, A, η, f) over $(\mathcal{N}, h)$ with compact base $\mathcal{N}$ s.t. the function $f \in C^\infty(\mathcal{M})$ is constant along the fibers. Let either*

(i) $\alpha \in \Omega^1(\mathcal{N})$ be a h-parallel 1-form with $\eta(\alpha^\sharp) = 0$ or

(ii) $\mathcal{N} = \mathcal{B} \times \mathbb{S}^1$, $\eta = du$ the coordinate 1-form[2] on $\mathbb{S}^1$, $b_1(\mathcal{B}) = 0$ and α a closed 1-form on $\mathcal{B}$.

Then, choosing $\Psi := \alpha \wedge \eta$, the manifold $(\mathcal{M}^{(n+2)}, g)$ is complete.

Proof. Let $K \in \Gamma(T\mathcal{M})$ as in the proof before. In this case, we have that

$$\mathscr{L}_K g = \frac{1}{2}\zeta(f)\pi^*\eta \odot \pi^*\eta + 2\pi^*\alpha \odot \pi^*\eta.$$

Assume there is an inextensible geodesic $\gamma : [0, \varepsilon) \longrightarrow \mathcal{M}$. To prove *(ii)*, let $\alpha = d\tau$. If we denote by $\delta := \mathrm{pr}_{\mathcal{B}} \circ \pi \circ \gamma$ the projected curve on $\mathcal{B}$, then by Lemma 4.2, δ is a $h_{\mathcal{B}}$-geodesic and as $\tau \in C^\infty(\mathcal{B})$,

$$\pi^*\alpha(\dot\gamma) = d\tau(d\pi(\dot\gamma)) = d\tau(\dot\delta) = h_{\mathcal{B}}(\mathrm{grad}_{h_{\mathcal{B}}} \tau, \dot\delta) \leq ||\,\mathrm{grad}_{h_{\mathcal{B}}} \tau|| \cdot ||\dot\delta||$$

by the Cauchy-Schwarz inequality. Hence, $\pi^*\alpha(\dot\gamma)$ is bounded. Since $\mathcal{N}$ and $\mathcal{M}$ are compact, $\zeta(f)$ is bounded, while $\pi^*\eta = -g(\xi, \cdot)$ is ∇^g-parallel. Hence, $(\mathscr{L}_K g)(\dot\gamma, \dot\gamma)$ is bounded and the assertion now follows from Proposition 4.4.

For case *(i)*, by $\nabla^h \alpha^\sharp = 0$ and the formula in Lemma 4.2 (ii), we see that

$$\frac{d}{dt}\pi^*\alpha(\dot\gamma) = \frac{d}{dt} g(\pi^*\alpha^\sharp, \dot\gamma(t)) = (2\alpha(\alpha^\sharp)\eta(\eta^\sharp) + \tfrac{1}{2} df(\alpha^\sharp) + 1) \cdot \pi^*\eta(\dot\gamma).$$

Since $\eta(\eta^\sharp), \alpha(\alpha^\sharp)$ and $df(\alpha^\sharp)$ are bounded $\frac{d}{dt}\alpha(\dot\gamma)$ is bounded on $[0, \varepsilon)$ and hence so is again $\pi^*\alpha(\dot\gamma)$. The same arguments as in *(i)* complete the proof. □

As we have already mentioned, the Lorentzian manifolds of type (Ψ, A, η, f) over $(\mathcal{N}, h)$ were already studied in [Lär11] to produce Lorentzian manifolds with special holonomy. Namely, there it was proven that for particular choices of (Ψ, η, f) and the base manifold $(\mathcal{N}, h)$, the resulting manifolds $(\mathcal{M}^{(n+2)}, g)$ of type (Ψ, A, η, f) over $(\mathcal{N}, h)$ have full holonomy $\mathrm{Hol}(\mathcal{M}^{(n+2)}, g) = (\mathbb{R}^+ \times G) \ltimes \mathbb{R}^n$ for recurrent or $\mathrm{Hol}(\mathcal{M}^{(n+2)}, g) = G \ltimes \mathbb{R}^n$ for parallel fundamental vector field $\xi \in \Gamma(T\mathcal{M})$, where $G := \mathrm{Hol}(\mathcal{B}, h_{\mathcal{B}})$ is the holonomy group of a certain Riemannian manifold $(\mathcal{B}, h_{\mathcal{B}})$ and $\mathcal{N} = \mathcal{B} \times \mathbb{S}^1$.[3] In particular, in [Lär11, Prop. 2.42] it is proven that taking $\mathcal{N} = \mathbb{T}^n = \mathbb{T}^{n-1} \times \mathbb{S}^1$ and $\Psi = du \wedge dv$, the resulting manifold $(\mathcal{M}^{(n+2)}, g)$ of type $(du \wedge dv, A, du, f)$ over $(\mathbb{T}^n, g_{\mathbb{T}^n})$ is complete. This result however turns out to be a special case of our Proposition 4.6. Moreover, all provided compact examples with special holonomy and base $\mathcal{N} = \mathcal{B} \times \mathbb{S}^1$ in [Lär11] are complete by Theorem 4.5, when $f \in C^\infty(\mathcal{M})$ is chosen to be constant along the fibers.

2 On $\mathbb{S}^1$ we consider the coordinate vector fields $\frac{\partial}{\partial\theta}$ resp. 1-forms $d\theta$ induced by the coordinates constituted by the *angle function* $\theta : \mathcal{U} \subset \mathbb{S}^1 \longrightarrow \mathbb{R}$ with $e^{i\theta(p)} = p$ for all $p \in \mathcal{U}$.

3 In [Lär11], the manifold $(\mathcal{M}^{(n+2)}, g)$ then is called of *toric type*.

4.3 GEOMETRY

A possible question in the discussed construction is, whether the obtained Lorentzian manifolds with special holonomy produce examples with certain distinguished geometries. Indeed, for the case $\mathcal{M} = \mathbb{R}^{n+2}$, a similar family of metrics was studied in [GP08, LG10], where Gibbons, Pope, Leistner and Galaev considered conditions under which certain Walker metrics produce Einstein metrics. An in some sense generalized but global version of the Walker metrics they considered is the presented construction of Lorentzian manifolds $(\mathcal{M}^{(n+2)}, g)$ of type (Ψ, A, η, f) over $(\mathcal{N}, h)$. The present section therefore deals with the question, whether these constructions produce Ricci-flat or even Einstein metrics with non-zero cosmological constant. As it turns out, the former is possible, while the latter is not due to the fact that the Hessian of $f \in C^\infty(\mathcal{M})$ cannot be constant on $\xi \times \xi$. Together with the former considerations in this paper we thus additionally obtain completeness results for the obtained Ricci-flat Lorentzian manifolds.

We proceed to present the formulas for the Riemannian curvature tensor $\mathcal{R}^g$ and the Ricci tensor Ric^g, where we use the sign convention

$$\mathcal{R}^g(X,Y)Z := \nabla^g_X \nabla^g_Y Z - \nabla^g_Y \nabla^g_X Z - \nabla^g_{[X,Y]} Z.$$

With the symbol $\owedge$ we denote the *Kulkarni–Nomizu product*.

Lemma 4.7. *Let $(\mathcal{M}^{(n+2)}, g)$ be of type (Ψ, A, η, f) over $(\mathcal{N}, h)$. Then the only non-vanishing terms of $\mathcal{R}^g$ are the following:*

$$\begin{aligned}
\mathcal{R}^g_{ijk\ell} &= \mathcal{R}^h_{ijk\ell} + \tfrac{1}{\|\eta^\sharp\|^2}(\nabla^h\eta \owedge \nabla^h\eta)(E_i, E_j, E_k, E_\ell), && (4.5)\\
\mathcal{R}^g_{i++j} &= \mathcal{R}^h_{i\eta\eta j} + \tfrac{1}{\|\eta^\sharp\|^2}(\nabla^h\eta \owedge \nabla^h\eta)(E_i, E_\eta, E_\eta, E_j)\\
&\quad + 2(\Psi(\cdot, E_\eta) \odot (\nabla^h\eta)(E_\eta))(E_i, E_j) + (\nabla^h_{E_i}\Psi)(E_\eta, E_j) + (\nabla^h_{E_j}\Psi)(E_\eta, E_i)\\
&\quad + h(\overline{\psi(E_i)}, \overline{\psi(E_j)}) - \tfrac{1}{2}(\mathrm{Hess}_g f)(\mathrm{e}_i, \mathrm{e}_j), && (4.6)\\
\mathcal{R}^g_{ijk+} &= \mathcal{R}^h_{ijk\eta} + \tfrac{1}{\|\eta^\sharp\|^2}(\nabla^h\eta \owedge \nabla^h\eta)(E_i, E_j, E_k, E_\eta)\\
&\quad + (\Psi(\cdot, E_\eta) \wedge (\nabla^h\eta)(E_k))(E_i, E_j) + (\nabla^h_{E_k}\Psi)(E_i, E_j), && (4.7)\\
\mathcal{R}^g_{i++\xi} &= -\tfrac{1}{2}(\mathrm{Hess}_g f)(\mathrm{e}_i, \xi), && (4.8)\\
\mathcal{R}^g_{+\xi\xi+} &= -\tfrac{1}{2}(\mathrm{Hess}_g f)(\xi, \xi). && (4.9)
\end{aligned}$$

Proof. The proof is straightforward by Lemma 4.2. Note that the $(2,0)$-tensor $\nabla\eta$ is symmetric since η is closed. Namely, as $0 = d\eta(X,Y) = X(\eta(Y)) - Y(\eta(X)) - \eta([X,Y])$, one infers

$$(\nabla^h_X\eta)(Y) = X(\eta(Y)) - \eta(\nabla^h_X Y) \overset{d\eta=0}{=} Y(\eta(X)) + \eta([X,Y]) - \eta(\nabla^h_X Y) = (\nabla^h_Y\eta)(X),$$

which justifies the term $(\nabla^h\eta \owedge \nabla^h\eta)$. Moreover, Ψ satisfies the second Bianchi identity since it is closed. □

As a corollary we obtain the following example of a compact Lorentzian pp-wave admitting *no horizontal and involutive realization* of the screen bundle.

Example 4.8. *Let $\mathcal{N} = \mathbb{T}^n \times \mathbb{S}^1$ be equipped with the metric $h = h_{\mathbb{T}^n} \oplus du^2$, where $h_{\mathbb{T}^n}$ is the flat metric on the torus constituted by the canonical coframe $\xi^1, \ldots, \xi^n$ on $\mathbb{T}^n$ and du the canonical 1-form on $\mathbb{S}^1$. Let $\Psi := \sum_{i,h=1}^n \psi_{ij}\xi^i \wedge \xi^j$ for constants $\psi_{ij} \in \mathbb{R}$ with $\psi_{ij} = -\psi_{ji}$ such that $0 \neq [\Psi] \in H^2(\mathbb{T}^n, \mathbb{Z}) \cap H^2_{\mathrm{dR}}(\mathbb{T}^n)$ and $\eta := du$. Then for any $f \in C^\infty(\mathcal{M})$ constant along the fibers, the manifold $(\mathcal{M}^{(n+2)}, g)$ of type (Ψ, A, η, f) over $(\mathbb{T}^n \times \mathbb{S}^1, h)$ possesses no horizontal and involutive realization of the screen bundle.*

Proof. Let $\pi : \mathcal{M}' \longrightarrow \mathbb{T}^n$ denote the circle bundle corresponding to $[\Psi]$ and hence $\mathcal{M} = \mathcal{M}' \times \mathbb{S}^1$. Let $\partial_1, \ldots, \partial_n$ be the canonical frame on $\mathbb{T}^n$ and $\xi^1, \ldots, \xi^n$ the canonical coframe on $\mathbb{T}^n$. Define $E_i := \partial_i$, $i = 1, \ldots, n$. By Lemma 4.3, the screen distribution $\mathbb{S} = \{\xi, Z\}^{\perp_g}$ defined by $Z = \frac{1}{2}\xi - \mathrm{e}_-$ is not involutive since $[\Psi] \neq 0$. Moreover, $(\mathcal{M}, g)$ is a pp-wave by Lemma 4.7 since $\nabla^h \Psi = 0$.

Now assume that $\widehat{\mathbb{S}} = \mathrm{span}(\widehat{S}_1, \ldots, \widehat{S}_n)$ is any other screen distribution defined by smooth functions b_i on $\mathcal{M}$ via $\widehat{S}_i = \mathrm{e}_i - b_i\xi$. Then

$$\nabla^g_X \widehat{S}_i = ((\pi^*\Psi)(X, \mathrm{e}_i) - db_i(X))\xi \tag{4.10}$$

for all $X \in \Gamma(\mathbb{L}^\perp)$. Let $\alpha^i := (\pi^*\Psi)(\cdot, \mathrm{e}_i)$ and assume that $\widehat{\mathbb{S}}$ is both, horizontal *and* involutive. By Proposition 1.19 this is the case if and only if $(d\alpha^i - db_i) = 0$ on $\mathbb{L}^\perp \cong T\mathcal{M}'$. In particular we would have $db_i(\xi) = 0$, i.e. the smooth functions b_i are constant along the fibers of π and thus descend to smooth functions φ_i on $\mathbb{T}^n$ (e.g. set $\varphi_i(x) = b_i(y, 1)$ for any $y \in \pi^{-1}(x)$). By (4.10) and horizontality of $\widehat{\mathbb{S}}$ we obtain

$$0 = \mathrm{i}F^A(\mathrm{e}_i, \mathrm{e}_j) - (db_j(\mathrm{e}_i) - db_i(\mathrm{e}_j)),$$

on $\mathcal{M}$, or equivalently, for the 1-form $\beta = \sum_{i=1}^n \varphi_i \xi^i$ on $\mathbb{T}^n$ that $\Psi(\partial_i, \partial_j) = d\beta(\partial_i, \partial_j)$. Hence $\Psi = d\beta$ which contradicts $[\Psi] \neq 0$. $\square$

To make notation short, we define the symmetric tensor $T_\eta : \Gamma(T\mathcal{M}) \times \Gamma(T\mathcal{M}) \longrightarrow \mathbb{R}$ as follows:

$$T_\eta(X, Y) := \frac{1}{||\eta^\sharp||^2} \sum_{k=1}^n (\nabla^h \eta \owedge \nabla^h \eta)(d\pi(X), \mathrm{e}_k, \mathrm{e}_k, d\pi(Y)). \tag{4.11}$$

By contraction of $\mathcal{R}^g$ we infer the non-vanishing terms of the Ricci tensor.

Lemma 4.9. *Let $(\mathcal{M}^{(n+2)}, g)$ be of type (Ψ, A, η, f) over $(\mathcal{N}, h)$. Then the only non-vanishing terms of Ric^g are the following:*

$$\mathrm{Ric}^g_{ij} = \mathrm{Ric}^h_{ij} + T_\eta(\mathrm{e}_i, \mathrm{e}_j), \tag{4.12}$$

$$\begin{aligned}\mathrm{Ric}^g_{i+} = \ & \mathrm{Ric}^h_{i\eta} + T_\eta(\mathrm{e}_i, \mathrm{e}_+) + \tfrac{\mathrm{i}}{2}((\mathrm{div}_g F^A)(\mathrm{e}_i) + F^A(\mathrm{e}_i, \zeta)\, \mathrm{div}_g \pi^*\eta) \\ & - \Psi(\nabla^h \eta^\sharp, E_\eta) - \tfrac{1}{2}(\mathrm{Hess}_g f)(\mathrm{e}_i, \xi),\end{aligned} \tag{4.13}$$

$$\begin{aligned}\mathrm{Ric}^g_{++} = \ & \mathrm{Ric}^h_{\eta\eta} + T_\eta(\mathrm{e}_+, \mathrm{e}_+) + 2\,\mathrm{trace}_h[\Psi(\cdot, E_\eta) \odot (\nabla^h \eta)(E_\eta)] \\ & - 2(\mathrm{div}_h \Psi)(E_\eta) + ||\overline{\psi}||_h - \tfrac{1}{2}\Delta_g f - \xi(f)(\tfrac{1}{2}\xi(f) + 1),\end{aligned} \tag{4.14}$$

$$\mathrm{Ric}^g_{\xi+} = -\tfrac{1}{2}(\mathrm{Hess}_g f)(\xi, \xi), \tag{4.15}$$

where div *is the divergence of a tensor.*[4]

4 Let T be a $(r, 0)$ tensor and g_0 a semi-Riemannian metric. Then we define the divergence of T through $\mathrm{div}_{g_0} T := \sum_k \varepsilon_k (\nabla^{g_0}_{e_k} T)(e_k, \cdot, \ldots, \cdot)$, where e_i is a g_0-orthonormal frame with $\varepsilon_k := g_0(e_k, e_k)$.

Remark 4.10. *If $\eta \in \Omega^1(\mathcal{N})$ is recurrent, i.e. $\nabla^h \eta = \alpha \otimes \eta$ for some $\alpha \in \Omega^1(\mathcal{N})$, then $(\nabla^h \eta \owedge \nabla^h \eta)$ and hence T_η already vanishes identically.*

Remark 4.11. *To compare the curvature equations of Lemma 4.7 and Lemma 4.9 with the results in [GP08], note that in their notation, $F_{\alpha\beta} = -F^A_{\alpha\beta}$ and $g_{\alpha\beta} = \delta_{\alpha\beta} \equiv const.$*

As the following theorem proves, this construction yields examples for Ricci-flat manifolds, even in the non-trivial case where $(\mathcal{N}, h)$ is Ricci-flat but $\Psi \neq 0$. An obvious obstruction is the fact that for g to be Ricci-flat, $f \in C^\infty(\mathcal{M})$ must be constant along the fibers due to (4.15).

Theorem 4.12. *Let $\mathcal{N} := \mathcal{B} \times \mathbb{S}^1$ or $\mathcal{N} := \mathcal{B} \times \mathbb{R}$ with $h := h_{\mathcal{B}} \oplus du^2$ for an n-dimensional Riemannian manifold $(\mathcal{B}, h_{\mathcal{B}})$. Moreover, let $(\mathcal{B}, h_{\mathcal{B}})$ be Ricci-flat and $\eta := du$ the coordinate 1-form on $\mathbb{S}^1$ resp. $\mathbb{R}$. Choose $\omega \in H^1_{\mathrm{dR}}(\mathcal{B}) \cap H^1(\mathcal{B}, \mathbb{Z})$ and a representative $\alpha \in \omega$ and consider the $\mathbb{S}^1$-bundle $\pi : \mathcal{M} \longrightarrow \mathcal{N}$ with $c_1(\mathcal{M}) = [\alpha \wedge \eta]$. Finally, choose $\Psi := \alpha \wedge \eta$ and $f := \widehat{f} \circ \pi \in C^\infty(\mathcal{M})$, where $\widehat{f} := f_{\mathcal{B}} \cdot f_{\mathbb{S}^1}$ with $f_{\mathcal{B}} \in C^\infty(\mathcal{B})$ and $f_{\mathbb{S}^1} \in C^\infty(\mathbb{S}^1)$.*

Then, the Lorentzian manifold $(\mathcal{M}^{(n+2)}, g)$ of type (Ψ, A, η, f) over $(\mathcal{N}, h)$ is Ricci-flat if and only if $\Delta_{h_{\mathcal{B}}}(f_{\mathcal{B}}) = -4\,\mathrm{div}_{h_{\mathcal{B}}}(\alpha)$.

Proof. Due to the definition of h and η, $\nabla^h \eta = 0$. As α and η are linearly independent, we may choose, locally, on $\mathcal{B}$ a local orthonormal frame $E_1, \ldots, E_n$ and consider the corresponding basis as in (4.2). Therefore, $F^A(\mathrm{e}_i, \mathrm{e}_j) = 0$ and $\Psi(E_i, E_j) = 0$ for all $i, j = 1, \ldots, n$. We obtain:

$$(\mathrm{div}_h \Psi)(E_\eta) = \mathrm{div}_{h_{\mathcal{B}}}(\alpha) = \mathrm{div}_{h_{\mathcal{B}}}(\alpha^\sharp).$$

As $(\mathcal{B}, h_{\mathcal{B}})$ is Ricci-flat, (4.14) turns into

$$\mathrm{Ric}^g_{++} = -\frac{1}{2}\Delta_{h_{\mathcal{B}}}(f_{\mathcal{B}}) - 2\,\mathrm{div}_{h_{\mathcal{B}}}(\alpha),$$

which proves the theorem. □

For the existence of concrete examples one needs to find solutions of the Poisson equation

$$\Delta_{h_{\mathcal{B}}}(f_{\mathcal{B}}) = -4\,\mathrm{div}_{h_{\mathcal{B}}}(\alpha).$$

Indeed, since $(\mathcal{B}, h_{\mathcal{B}})$ is assumed to be connected and without boundary, we obtain the following:

Corollary 4.13. *If $(\mathcal{B}, h_{\mathcal{B}})$ is a compact Ricci-flat manifold then we always find a unique (up to a constant) $f_{\mathcal{B}} \in C^\infty(\mathcal{B})$ s.t. the Lorentzian manifold $(\mathcal{M}^{(n+2)}, g)$ as in Theorem 4.12 with $\mathcal{N} = \mathcal{B} \times \mathbb{S}^1$ is Ricci-flat.*

Proof. If $\omega = 0$ we choose $f_{\mathcal{B}}$ such that $\alpha = -\frac{1}{4} d f_{\mathcal{B}}$ with $\alpha \in \omega$. Otherwise, since $\mathrm{div}(\alpha) = \mathrm{div}(\alpha^\sharp)$ and $\int_{\mathcal{B}} \mathrm{div}(\alpha^\sharp) = 0$ as $\partial\mathcal{B} = \emptyset$, we always find a unique (up to a constant) solution $f_{\mathcal{B}} \in C^\infty(\mathcal{B})$ to the Poisson equation $\Delta^{h_{\mathcal{B}}}(f_{\mathcal{B}}) = -4\,\mathrm{div}_{h_{\mathcal{B}}}(\alpha)$, cf. [Aub98, Theorem 4.7]. □

If $(\mathcal{B}, h_\mathcal{B})$ is compact with $b_1(\mathcal{B}) > 0$ and $\omega \neq 0$, the representative $\alpha \in \omega$ needs to be chosen non-harmonic for the function $f_\mathcal{B} \in C^\infty(\mathcal{B})$ to be non-constant. Moreover, note that the condition $b_1(\mathcal{B}) > 0$ is satisfied for compact Ricci-flat Riemannian manifolds whenever on $(\mathcal{B}, h_\mathcal{B})$ exists at least one Killing vector field, since in this case $b_1(\mathcal{B}) = \dim_\mathbb{R} \mathfrak{kill}(\mathcal{B}, h_\mathcal{B})$, cf. [Bes87, Theorem 1.84]. For example one may take the Ricci-flat metric on some Calabi-Yau manifold of dimension $n = 2m$, i.e. a compact Kähler manifold with trivial first Chern class. Examples for such are e.g. $\mathrm{K3} \times \mathbb{T}^k$ or more generally products of compact hyper-Kähler manifolds, i.e. a $4k$-dimensional Riemannian manifold with holonomy contained in $\mathrm{Sp}(k)$, with the flat torus. Another list of examples can be constructed from [FW75, Theorem 4.1]. For $b_1(\mathcal{B}) = 0$ we can take compact G_2- or $\mathrm{Spin}(7)$-manifolds [Joy96a, Joy96b] which are always Ricci flat. However, if $(\mathcal{B}, h_\mathcal{B})$ is a $\mathrm{Spin}(7)$-manifold it is also simply-connected an hence the constructed circle bundle in Theorem 4.12 is trivial.

Moreover, by Proposition 4.6, the compact manifolds in Corollary 4.13 and thus in particular the just stated examples, are all complete.

Corollary 4.14. *Every compact Ricci-flat Lorentzian manifold occurring in Corollary 4.13 is complete. This even holds for arbitrary $f_\mathcal{B} \in C^\infty(\mathcal{B})$.*

Proof. Choose $\omega \in H^1_{\mathrm{dR}}(\mathcal{B}) \cap H^1(\mathcal{B}, \mathbb{Z})$ and a representative $\alpha \in \omega$. If $\omega = 0$ then Proposition 4.6 (ii) proves the statement. When $\omega \neq 0$ and $(\mathcal{B}, h_\mathcal{B})$ is assumed to be compact and Ricci-flat, we can write $\alpha = \widehat{\alpha} + d\varphi$, where $\widehat{\alpha} = K^\flat$ is the dual 1-form to a Killing field

$$K \in \mathfrak{kill}(\mathcal{B}, h_\mathcal{B}) = \{X \in \Gamma(T\mathcal{M}) \mid \nabla^{h_\mathcal{B}} X = 0\}.$$

Let $\Psi := \alpha \wedge \eta$, $\widehat{\Psi} := \widehat{\alpha} \wedge \eta$ and A, $\widehat{A}$ denote corresponding connection forms, i.e. with $dA = -2\pi i \pi^* \Psi$ and $d\widehat{A} = -2\pi i \pi^* \widehat{\Psi}$, respectively. Then

$$A = \widehat{A} - 2\pi i (\varphi \circ \pi) \pi^* \eta. \tag{4.16}$$

With the data chosen as in Theorem 4.12 we infer

$$\begin{aligned} g &= 2iA \odot \pi^*\eta + (f+1) \cdot \pi^*\eta \odot \pi^*\eta + \pi^* h_\mathcal{B} \\ &\overset{(4.16)}{=} 2i\widehat{A} \odot \pi^*\eta + (4\pi(\varphi \circ \pi) + f) \cdot \pi^*\eta \odot \pi^*\eta + \pi^* h_\mathcal{B} \\ &= 2i\widehat{A} \odot \pi^*\eta + \widehat{f} \cdot \pi^*\eta \odot \pi^*\eta + \pi^* h_\mathcal{B} \end{aligned}$$

for $\widehat{f} := 4\pi(\varphi \circ \pi) + f$. Hence, the Lorentzian manifold $(\mathcal{M}^{(n+2)}, g)$ is of type $(\widehat{\Psi}, \widehat{A}, \eta, \widehat{f})$ over $(\mathcal{N}, h)$ and the assumptions of Proposition 4.6 (i) are all satisfied, yielding the completeness. □

For the Einstein case with non-zero cosmological constant and particular Ricci-flat cases one has the following non-existence result:

Proposition 4.15. *Let $(\mathcal{M}^{(n+2)}, g)$ be any Lorentzian manifold of type (Ψ, A, η, f) over $(\mathcal{N}, h)$. Then it holds:*

(i) $(\mathcal{M}^{(n+2)}, g)$ cannot be an Einstein manifold with non-zero cosmological constant.

(ii) Let $(\mathcal{M}^{(n+2)}, g)$ be Ricci-flat and $\mathcal{N}$ compact. If either

a) $\eta^\sharp$ is a h-Killing field, $\zeta(f) = 0$ and $\Psi(\eta^\sharp, \cdot) = 0$, or

b) $\eta^\sharp$ is h-parallel, $\mathcal{N} = \mathcal{B} \times \mathbb{S}^1$ and $\Psi \in \Omega^2(\mathcal{B})$,

then $\Psi \in \Omega^2(\mathcal{N})$ must already vanish identically.

Proof. To prove *(i)*, suppose the Lorentzian manifold $(\mathcal{M}^{(n+2)}, g)$ of type (Ψ, A, η, f) over $(\mathcal{N}, h)$ is an Einstein manifold. Then, by Lemma 4.9 (4.15), the cosmological constant Λ has to be equal to $\frac{1}{2}(\text{Hess}\, f)(\xi, \xi)$. Hence, $(\text{Hess}\, f)(\xi, \xi)$ has to be constant on each fiber since $0 = \xi(\Lambda) = \xi((\text{Hess}\, f)(\xi, \xi))$ implying $(\text{Hess}\, f)(\xi, \xi)|_{\pi^{-1}(y)} \equiv const$ for all $y \in \mathcal{N}$. As a consequence, such $f \in C^\infty(\mathcal{M})$ would give rise (by passing to a local trivialization) to a function $\widehat{f} \in C^\infty(\mathbb{S}^1)$ with constant Laplacian on $\mathbb{S}^1$. Hence, f is then contant on the fibers. But this is a contradiction to $\Lambda \neq 0$.

To see *(ii.a)* assume that $(\mathcal{M}^{(n+2)}, g)$ is Ricci-flat and the assumptions above hold true. Note that necessarily $\xi(f) = 0$. Then, by Proposition 4.5, there exists a timelike Killing vector field $K \in \Gamma(T\mathcal{M})$. Due to [RS96, Theorem 3.2], K then has to be parallel. This is the case if and only if Ψ vanishes, since $g(\nabla^g_{e_i} K, e_j) = \Psi_{ij}$ by Lemma 4.2 (iii).

In the case *(ii.b)*, Ricci-flatness of $(\mathcal{M}^{(n+2)}, g)$ implies $\Delta_h f = 2||\psi||_h$ by Lemma 4.9 (4.14), where we regard f as a function on $\mathcal{N}$ which is feasible since f is constant along the fibers. Since necessarily $\int_{\mathcal{N}} \Delta_h f = 0$ we infer $||\psi||_h = 0$ and hence $\Psi = 0$. □

Note that this proposition implies in particular, that the toric type constructions in [Lär11] with compact base $\mathcal{N} = \mathcal{B} \times \mathbb{S}^1$ and $(\mathcal{B}, h_{\mathcal{B}})$ being Ricci-flat or Einstein cannot produce Ricci-flat or Einstein metrics on $\mathcal{M}$ provided that $\Psi \in \Omega^2(\mathcal{B})$ is not chosen to be zero.

4.4 HOLONOMY

Within this section we intend to apply Theorem 1.12 to the manifolds occurring in Theorem 4.12 in order to determine their full holonomy. Moreover we will construct certain Lorentzian manifolds of type (Ψ, A, η, f) over $(\mathcal{N}, h)$ having holonomy of type 4, see Theorem 1.7.

Full Holonomy of Ricci Flat Examples

Indeed, we find for the universal cover $(\widetilde{\mathcal{M}}, \widetilde{g})$ of the Lorentzian manifolds of type (Ψ, A, η, f) over $(\mathcal{N}, h)$ appearing in Theorem 4.12 the following.

Proposition 4.16. *Let $(\mathcal{M}^{(n+2)}, g)$ be a Lorentzian manifold of type (Ψ, A, η, f) over $(\mathcal{N}, h)$ for a smooth function $f \in C^\infty(\mathcal{M})$ constant along the fibers and $\mathcal{N} = \mathcal{B} \times \mathbb{S}^1$ for a compact Riemannian manifold $(\mathcal{B}, h_{\mathcal{B}})$ with $b_1(\mathcal{B}) > 0$. Choose for η the coordinate 1-form on $\mathbb{S}^1$, $h = h_{\mathcal{B}} \oplus \eta^2$ and $\Psi := \alpha \wedge \eta$ for some nowhere vanishing closed 1-form α s.t. $[\alpha] \in H^1_{\text{dR}}(\mathcal{B}) \cap H^1(\mathcal{B}, \mathbb{Z})$. Then the universal cover $(\widetilde{\mathcal{M}}, \widetilde{g})$ is isometric to a manifold*

$$(\mathbb{R}^2 \times \mathcal{S},\ \Xi_{(u,v,p)} = 2dudv + \kappa(u,p)du^2 + A_u \odot du + \Theta_p), \tag{4.17}$$

with $A_u = 2(u + a(s))ds$, $a \in \mathbb{C}^\infty(\mathbb{R})$, and where s is the $\mathbb{R}$-coordinate of $\mathcal{S} = \mathbb{R} \times \mathcal{A}$ which is the universal cover of a leaf of the involutive screen distribution $\mathbb{S}|_{\mathcal{L}^\perp}$ defined in (4.3) *on page 68. Further, $\mathcal{L}^\perp$ is a leaf of $\mathbb{L}^\perp$, $\kappa : \widetilde{\mathcal{M}} \longrightarrow \mathbb{R}$ is a smooth function not depending on the v-coordinate and Θ is a Riemannian metric on $\mathcal{S}$ which coincides with the lift of $\pi^* h_{\mathcal{B}}$ to the universal cover, restricted to $\mathcal{S}$.*

Proof. To this end, let a tilde ahead of any object denote the lift to the universal cover. Moreover we will use, locally, a basis of $T\mathcal{B}$ of $h_{\mathcal{B}}$-orthonormal vector fields $E_\alpha, E_2, \ldots, E_n$ with $E_\alpha := \frac{\alpha^\sharp}{||\alpha^\sharp||}$ and $E_2, \ldots, E_n \in \ker \alpha$. As usual we write $\mathrm{e}_i := E_i^*$ and set $\mathrm{e}_\alpha := E_\alpha^*$.

We first show how to separate $\mathbb{R}^3$ from the universal cover $\widetilde{\mathcal{M}}$ using Proposition 1.44. Indeed, $\pi^*\eta$ is closed on $\mathcal{M}$ and $(\pi^*\eta)(\zeta) = 1$. Moreover, the 1-form $\pi^*\alpha$ on $\mathcal{M}$ is closed, too, and fulfills $(\pi^*\alpha)(\mathrm{e}_\alpha) = 1$. Finally, fix a leaf $\mathcal{L}^\perp$ of $\mathbb{L}^\perp$. Since $\mathbb{S}$ is horizontal and involutive, cf. Lemma 4.3, the 1-form $\zeta^\flat = g(\zeta, \cdot) = \mathrm{i}A + (f+1)\pi^*\eta$ is closed on $\mathcal{L}^\perp$. We can now apply Proposition 1.44 three times:

$$\widetilde{\mathcal{M}} \overset{\pi^*\eta}{\simeq} \mathbb{R} \times \widetilde{\mathcal{L}^\perp} \overset{\zeta^\flat}{\simeq} \mathbb{R} \times \mathbb{R} \times \mathcal{S} \overset{\pi^*\alpha}{\simeq} \mathbb{R}^3 \times \mathcal{A},$$

where $\mathcal{S}$ is a fixed leaf of $\widetilde{\mathbb{S}}|_{\widetilde{\mathcal{L}^\perp}}$ and $\mathcal{A}$ is a leaf of $\ker \widetilde{\pi^*\alpha}|_{\mathcal{S}}$. Recall that the diffeomorphisms are given by the flows of $\widetilde{\zeta}$, $-\widetilde{\xi}$ and $\widetilde{\mathrm{e}_\alpha}$, respectively. To be more precise, let $\{\varphi_u^\eta\}_{u\in\mathbb{R}}$, $\{\varphi_v^\xi\}_{v\in\mathbb{R}}$ and $\{\varphi_s^\alpha\}_{s\in\mathbb{R}}$ denote the corresponding flows of the latter vector fields, respectively. Then

$$\Phi : \mathbb{R}^3 \times \mathcal{A} \ni (u, v, s, p) \longmapsto \varphi_u^\eta(\varphi_v^\xi(\varphi_s^\alpha(p))) \in \widetilde{\mathcal{M}}$$

is the asserted diffeomorphism.

Since all vector fields except ζ and e_α commute, we obtain

$$d\Phi(\partial_v) = -\widetilde{\xi},\ d\Phi(\partial_u) = \widetilde{\zeta}.$$

As $\mathscr{L}_\zeta(\pi^*\alpha) = 0$, the flow of $\widetilde{\zeta}$ preserves $\widetilde{\pi^*\alpha}$ and thus

$$\widetilde{g}(d\Phi(\partial_s), \widetilde{\mathrm{e}_\alpha}) = \widetilde{g}(d\varphi_u^\eta(\widetilde{\mathrm{e}_\alpha}), \widetilde{\mathrm{e}_\alpha}) = [(\varphi_u^\eta)^* \widetilde{\pi^*\alpha}](\widetilde{\mathrm{e}_\alpha}) = 1.$$

Moreover let, locally, $\omega_j := g(\mathrm{e}_j, \cdot)$, $j = 2, \ldots, n$. Then $\mathscr{L}_\zeta \omega_j = 0$ and hence

$$\widetilde{g}(d\Phi(\partial_s), \widetilde{\mathrm{e}_j}) = \widetilde{g}(d\varphi_u^\eta(\widetilde{\mathrm{e}_\alpha}), \mathrm{e}_j) = (\varphi_u^\eta)^* \widetilde{\omega_j}(\widetilde{\mathrm{e}_\alpha}) = \widetilde{\omega_j}(\widetilde{\mathrm{e}_\alpha}) = \omega_j(\mathrm{e}_\alpha) = 0.$$

Therefore, we obtain that

$$d\Phi(\partial_s) = \widetilde{\mathrm{e}_\alpha} + \tau \cdot \widetilde{\xi}$$

for some $\tau \in C^\infty(\widetilde{\mathcal{M}})$. Since $d\pi^*\alpha = 0$ we obtain $\mathscr{L}_{\mathrm{e}_\alpha}(\pi^*\alpha) = 0$. Hence, every flow defining Φ preserves $\pi^*\alpha$ and since, locally, $\mathbb{S} = \mathrm{span}\{\mathrm{e}_2, \ldots, \mathrm{e}_n\}$, we see that

$$d\Phi(\widetilde{\mathrm{e}_i}) \in \Gamma(\widetilde{\ker \pi^*\alpha}), \quad i = 2, \ldots, n.$$

Since $\pi^*\Psi \in \Omega^2(\mathcal{M})$ is closed, its lift to the universal cover is exact. More precisely we have

$$\Phi^* \widetilde{\pi^*\Psi} = \Phi^* \widetilde{\pi^*\alpha} \wedge \Phi^* \widetilde{\pi^*\eta} = ds \wedge du$$

as $\Phi^*\widetilde{\pi^*\eta} = du$ and $\Phi^*\widetilde{\pi^*\alpha} = ds$. Hence,

$$\mathrm{i}\Phi^* d\widetilde{A} = \mathrm{i}\Phi^*\widetilde{F^A} = 2\Phi^*\widetilde{\pi^*\Psi} = 2ds \wedge du. \tag{4.18}$$

Using this together with $\widetilde{\mathrm{i}A}(d\Phi(\partial_v)) = -\mathrm{i}A(\xi) = 1$ and $\widetilde{\mathrm{i}A}(d\Phi(\partial_s)) = \tau \cdot \mathrm{i}A(\xi) = -\tau$, we see that

$$\Phi^*(\widetilde{\mathrm{i}A}) = dv - \tau ds$$

and hence $d\tau = -2du - b(s)ds$ by (4.18), whence $\tau = -2(a(s) + u)$ for $2\frac{d}{ds}a = b$. Summarizing we get:

$$\begin{aligned}(\Phi^*\widetilde{g}) &= 2\Phi^*(\widetilde{\mathrm{i}A}) \odot \Phi^*\widetilde{\pi^*\eta} + (\widetilde{f}\circ\Phi + 1)(\Phi^*\widetilde{\pi^*\eta})^2 + \Phi^*(\widetilde{\pi^*h_{\mathcal{B}}})\\ &= 2(dv + 2(u + a(s))ds + (\widetilde{f}\circ\Phi + 1)du)du + \Phi^*(\widetilde{\pi^*h_{\mathcal{B}}})\\ &= 2dudv + \kappa du^2 + A_u \odot du + \Theta,\end{aligned}$$

where $\Theta := \Phi^*(\widetilde{\pi^*h_{\mathcal{B}}})$ and $\kappa := \widetilde{f}\circ\Phi + 1$, while $\partial_v\kappa = 0$ since $\xi(f) = 0$, i.e. κ is independent of the v-coordinate. □

Next we need a description of the fundamental group of $\mathcal{M}$ since this is contained in the groups Q of Theorem 1.12. Using Serre's long exact sequence for the $\mathbb{S}^1$-bundle $\pi : \mathcal{M} \longrightarrow \mathcal{N}$ with $\mathcal{N} = \mathcal{B}\times\mathbb{S}^1$ we obtain

$$0 \to \pi_2(\mathcal{M}) \xrightarrow{\varphi_1} \pi_2(\mathcal{B}) = \pi_2(\mathcal{N}) \xrightarrow{\varphi_2} \pi_1(\mathbb{S}^1) = \mathbb{Z} \xrightarrow{\varphi_3} \pi_1(\mathcal{M}) \xrightarrow{\varphi_4} \pi_1(\mathcal{B})\times\mathbb{Z} \to 0. \tag{4.19}$$

This can be rewritten as the two short exact sequences

$$0 \longrightarrow \pi_2(\mathcal{M}) \xrightarrow{\varphi_1} \pi_2(\mathcal{B}) \xrightarrow{\varphi_2} \operatorname{im}\varphi_2 \longrightarrow 0 \tag{4.20}$$

$$0 \longrightarrow \operatorname{coker}\varphi_2 \xrightarrow{\varphi_3} \pi_1(\mathcal{M}) \xrightarrow{\varphi_4} \pi_1(\mathcal{B})\times\mathbb{Z} \longrightarrow 0 \tag{4.21}$$

To determine $\pi_1(\mathcal{M})$ from (4.21), we make the following definition.

Definition 4.17. *We say that $\pi_1(\mathcal{B})$* **is split**, *iff the short exact sequence* (4.21) *splits.*

For example, $\pi_1(\mathcal{B})$ is split, if it is a free group. We obtain:

Proposition 4.18. *If $\pi_1(\mathcal{B})$ is split then* $\pi_1(\mathcal{M}) \cong (\pi_1(\mathcal{B})\times\mathbb{Z}) \ltimes \operatorname{coker}\varphi_2 = (\pi_1(\mathcal{B})\times\mathbb{Z}) \ltimes \mathbb{Z}/\operatorname{im}\varphi_2$.

Since every subgroup of a free group is free, so is $\operatorname{im}\varphi_2 \subset \mathbb{Z}$ and consequently the sequence (4.20) always splits and gives us a possibility to calculate either $\pi_2(\mathcal{B})$ or $\pi_2(\mathcal{M})$:

Proposition 4.19. $\pi_2(\mathcal{B}) \cong \operatorname{im}\varphi_2 \ltimes \pi_2(\mathcal{M})$.

For example, in the easiest case where $\pi_2(\mathcal{B}) = 0$ (e.g. when a cover of $\mathcal{B}$ is contractible), then $\operatorname{im}\varphi_2 = 0$ and hence $\pi_1(\mathcal{M}) = (\pi_1(\mathcal{B})\times\mathbb{Z}) \ltimes \mathbb{Z}$ by Proposition 4.18. If, for instance $\pi_2(\mathcal{B}) = \mathbb{Z}$ (e.g. when $\mathcal{B} = \mathbb{CP}^n$), then $\operatorname{coker}\varphi_2 \in \{1, \mathbb{Z}/k\mathbb{Z}, \mathbb{Z}\}$ and Propsition 4.19 may help to determine the correct case if one is able to get information about $\pi_2(\mathcal{M})$. For instance, if the leaves of $\mathbb{L}^\perp = \xi^\perp$ are compact, then $\mathcal{M}$ fibers over $\mathbb{S}^1$

with each fiber diffeomorphic to a leaf $\mathcal{L}^\perp$ by Theorem 1.24 (iii) and Serre's long exact sequence yields $\pi_2(\mathcal{M}) \cong \pi_2(\mathcal{L}^\perp)$ and $\pi_1(\mathcal{M}) = \mathbb{Z} \ltimes \pi_1(\mathcal{L}^\perp)$.

We are now in the position to use Theorem 1.12 to give a description of the holonomy of the Lorentzian manifolds of type (Ψ, A, η, f) over $(\mathcal{N}, h)$ considered in Proposition 4.16. We obtain:

Theorem 4.20. *Let $(\mathcal{M}^{(n+2)}, g)$ be a Lorentzian manifold of type (Ψ, A, η, f) over $(\mathcal{N}, h)$ with the data chosen as in Proposition 4.16 with $f \in C^\infty(\mathcal{N})$ s.t. $\mathrm{Hess}_B f|_{\mathcal{B}}$ is non-degenerate in a point. Then the full holonomy group is given by*

$$\mathrm{Hol}_x(\mathcal{M}^{(n+2)}, g) = O \cdot \mathrm{Hol}^0_q(\mathcal{B}, h_{\mathcal{B}}) \ltimes \mathbb{R}^n, \tag{4.22}$$

where $(\mathrm{pr}_{\mathcal{B}} \circ \pi \circ \Phi)(\widetilde{x}) = q$, $\widetilde{x} = (u, v, p)$, $\Phi(\widetilde{x}) = x$ and

$$O := \left\langle (d\mu_{\sigma^{-1}})^{-1} \circ \mathcal{P}^\Theta_\sigma \mid \sigma \in \pi_1(\mathcal{M}) \right\rangle \subset \mathrm{O}(n),$$

with the notations as in Theorem 1.12. Moreover, we can replace $\pi_1(\mathcal{M})$ by $\pi_1(\mathcal{B})$ in O, if $\pi_1(\mathcal{B})$ is split. In this case we actually have

$$\mathrm{Hol}_x(\mathcal{M}^{(n+2)}, g) = \mathrm{Hol}_q(\mathcal{B}, h_{\mathcal{B}}) \ltimes \mathbb{R}^n. \tag{4.23}$$

Proof. The proof of the theorem is threefold. As a first step we show that the manifolds occurring in Proposition 4.16 have full holonomy $\mathrm{Hol}(\mathcal{S}, \Theta) \ltimes \mathbb{R}^n$ which is an easy adaption of the proof of Theorem 1.11, see [BLL14, Proposition 4]. In a second step we prove that $\mathrm{Hol}(\mathcal{S}, \Theta)$ is isomorphic to $\mathrm{Hol}^0(\mathcal{B}, h_{\mathcal{B}})$. Finally, we provide the arguments for the missing $\mathbb{R}^*$-factor in the groups Q occurring in Theorem 1.12 and the fact that it suffices to consider generators $\sigma \in \pi_1(\mathcal{B})$.

Step 1: We prove that for the $(n+2)$-dimensional manifold $\widetilde{\mathcal{M}} = \mathbb{R}^2 \times \mathcal{S}$ equipped with the metric $\Xi_{(u,v,p)} = 2dudv + \kappa(u,p)du^2 + A_u \odot du + \Theta_p$ with simply-connected $\mathcal{S} \simeq \mathbb{R} \times \mathcal{A}$, and $A_u = 2\mu ds := 2(u + a(s))ds$, the full holonomy in the point $\widetilde{x} = (0, 0, p)$ is given by

$$\mathrm{Hol}_{\widetilde{x}}(\widetilde{\mathcal{M}}, \Xi) = \mathrm{Hol}_p(\mathcal{S}, \Theta) \ltimes \mathbb{R}^n. \tag{4.24}$$

Here, $p \in \mathcal{S}$ is a point s.t. $(\mathrm{Hess}_\Theta \kappa)(p)$ is non-degenerate. To prove (4.24) observe that the only non-vanishing components of the Levi-Civita connection ∇ to Ξ are given by

$$\begin{aligned}
&\nabla_{\partial_u} Y = \tfrac{1}{2} d\kappa(Y)\partial_v, && \nabla_{\partial_u}\partial_u = (\tfrac{1}{2} d\kappa(\partial_u) + \tfrac{\mu}{2} d\kappa(\partial_s) - \mu)\partial_v + \partial_s - \tfrac{1}{2}\,\mathrm{grad}_\Theta \kappa, \\
&\nabla_{\partial_u}\partial_s = \tfrac{1}{2} d\kappa(\partial_s)\partial_v, && \nabla_{\partial_s} X = \nabla_X \partial_s = \nabla^\Theta_X \partial_s, \\
&\nabla_{\partial_s}\partial_s = \nabla^\Theta_{\partial_s}\partial_s + a'(s)\partial_v, && \nabla_X Y = \nabla^\Theta_X Y,
\end{aligned} \tag{4.25}$$

where $X, Y \in \Gamma(T\mathcal{A})$. Since the function a does not depend on the u-coordinate we get for the curvature R of Ξ

$$R(\partial_u, S_1)S_2 = -\frac{1}{2}\mathrm{Hess}_\Theta \kappa(S_1, S_2)\partial_v, \tag{4.26}$$

for all $S_1, S_2 \in \Gamma(T\mathcal{S})$. Hence, the holonomy algebra of $(\widetilde{\mathcal{M}}, \Xi)$ in $\widetilde{x}$ contains $\mathbb{R}^n$. Let $\gamma : [0,1] \longrightarrow \widetilde{\mathcal{M}}$ be a curve with $\gamma(t) = (u(t), v(t), s(t), \delta(t))$ with $\gamma(0) = (0, 0, p)$, $p = (s, q)$ and $\delta : [0,1] \longrightarrow \mathcal{A}$ a curve with $\delta(0) = q$. Then, for $X \in \Gamma(T\mathcal{S})$ being the

Θ-parallel vector field along $(s(t), \delta(t))$ with $X(0) = v \in T_p\mathcal{S}$, we obtain for the parallel displacement $\mathcal{P}$ w.r.t. Ξ that

$$\mathcal{P}_{\gamma|_{[0,t)}}(v) = \varphi_v(t) \cdot (\partial_v \circ \gamma(t)) + X(t)$$

with $\varphi_v : [0,1] \longrightarrow \mathbb{R}$ defined as

$$\varphi_v(t) = -\tfrac{1}{2} \int_0^t \Big(\dot{u}(r) d\kappa_{\gamma(r)}(X(\gamma(r))) + \rho(r) \Big) \, \mathrm{d}r,$$

where

$$\rho(r) = \begin{cases} 2\dot{s}(r)a'(s(r)), & v \in \mathbb{R}\partial_s, \\ 0, & v \in T_q\mathcal{A}. \end{cases}$$

Therefore, $\mathrm{pr}_{T_p\mathcal{S}} \circ \mathcal{P}_\gamma|_{T_p\mathcal{S}} = \mathcal{P}^{\Theta}_{(s,\delta)}$ which proves (4.24).

Step 2: We are going to prove

$$\mathrm{Hol}^0_x(\overline{\mathcal{S}}, \overline{h}) \cong \mathrm{Hol}^0_q(\mathcal{B}, h_{\mathcal{B}}). \tag{4.27}$$

Here, $\overline{h} := \overline{\pi}^* h_{\mathcal{B}}|_{\overline{\mathcal{S}} \times \overline{\mathcal{S}}}$, where $\overline{\mathcal{S}}$ is a leaf of the involutive screen distribution $\mathbb{S}$ and $\overline{\pi}(x) = q$ with $x \in \overline{\mathcal{S}}$, where $\overline{\pi} : \overline{\mathcal{S}} \longrightarrow \mathcal{B}$ denotes the surjective map $\overline{\pi} := \mathrm{pr}_{\mathcal{B}} \circ \pi|_{\overline{\mathcal{S}}}$. Then (4.27) obviously implies $\mathrm{Hol}(\mathcal{S}, \Theta) \cong \mathrm{Hol}^0(\mathcal{B}, h_{\mathcal{B}})$.

First note that $(\overline{\mathcal{S}}, \overline{h})$ is geodesically complete since it is the restriction of a complete Riemannian metric g^R on $\mathcal{M}$ (namely, $g^R = -A \odot A + \zeta^\flat \odot \zeta^\flat + \overline{\pi}^* h_{\mathcal{B}}$) to a leaf (namely, $\overline{\mathcal{S}}$) of a foliation, cf. Proposition 1.26. In addition it holds $\overline{\pi}^* h_{\mathcal{B}} = \overline{h}$, i.e. $\overline{\pi}$ is a local isometry, and thus

$$d\overline{\pi}_x \circ \mathcal{P}^{\overline{h}}_{\overline{\gamma}} \circ d\overline{\pi}_x^{-1} = \mathcal{P}^{h_{\mathcal{B}}}_{\overline{\pi} \circ \overline{\gamma}} \tag{4.28}$$

for any loop $\overline{\gamma}$ in x. Finally, $\overline{\pi}$ is a Riemannian covering and hence every null-homotopic loop in $\mathcal{B}$ lifts to a null-homotopic loop in $\overline{\mathcal{S}}$ so (4.27) follows from (4.28).

Step 3: Let $\Phi : \widetilde{\mathcal{M}} \simeq \mathbb{R}^2 \times \mathcal{S} \longrightarrow \mathcal{M}$ denote the universal covering from Proposition 4.16 with $d\Phi(\partial_u) = \zeta$, $d\Phi(\partial_v) = -\xi$ and hence $\Phi^*\pi^*\eta = du$. When $\sigma \in \pi_1(\mathcal{M})$ is a deck transformation of $(\widetilde{\mathcal{M}}, \Xi = \Phi^* g)$, i.e. $\Phi \circ \sigma = \Phi$, then we see that

$$\sigma^* du = \sigma^*(\Phi^*(\pi^*\eta)) = (\Phi \circ \sigma)^* \pi^* \eta = \Phi^* \pi^* \eta = du.$$

Hence $u \circ \sigma = u + b_\sigma$, i.e. $a_\sigma = 1$ so there is no $\mathbb{R}^*$-factor in the groups Q occurring in Theorem 1.12.

Assume now that $\pi_1(\mathcal{B})$ is split, i.e. (4.21) splits, then $\pi_1(\mathcal{M}) \cong (\pi_1(\mathcal{B}) \times \mathbb{Z}) \ltimes \mathbb{Z}/\operatorname{im}\varphi_2$ by Proposition 4.18. Let $x_0 \in \mathcal{M}$. Then the integer factors in $\pi_1(\mathcal{M}, x_0)$ come from the fundamental groups of the fibers and the circle in $\mathcal{N}$. These are in turn generated by the flow of ξ and ζ starting in x_0, respectively. Hence, if $\widetilde{x_0} = (u, v, p) \in \widetilde{\mathcal{M}}$ with $\Phi(\widetilde{x_0}) = x_0$ and for $k \in \mathbb{Z}$

$$\widetilde{\gamma}^\zeta_k(t) := (u + kt, v, p), \; \widetilde{\gamma}^\xi_k(t) := (u, kt - v, p),$$

then $\Phi \circ \widetilde{\gamma}^\zeta_k$ and $\Phi \circ \widetilde{\gamma}^\xi_k(t)$ are generators for the integer factors in $\pi_1(\mathcal{M}, x_0)$ since it are integral curves of $k\zeta$ and $k\xi$, respectively. But neither $\widetilde{\gamma}^\zeta_k$ nor $\widetilde{\gamma}^\xi_k$ can connect $\widetilde{x_0}$ with $\sigma(\widetilde{x_0})$ for some isometry σ of $(\widetilde{\mathcal{M}}, \Xi)$ with $\nu(u, v, \cdot) \neq \mathrm{id}_{\mathcal{S}}$. So we can replace $\pi_1(\mathcal{M})$ by $\pi_1(\mathcal{B})$ in O. Since then $\mathrm{Hol}_q(\mathcal{B}, h_{\mathcal{B}}) = O \cdot \mathrm{Hol}^0_q(\mathcal{B}, h_{\mathcal{B}})$ by Proposition 1.2, this completes the proof. $\square$

If $b_1(\mathcal{B}) = 0$, we cannot choose a nowhere vanishing closed 1-form $\alpha \in \Omega^1(\mathcal{B})$ since it must be exact and hence $\alpha = d\tau$ for some smooth function $\tau \in C^\infty(\mathcal{B})$. But as $\mathcal{B}$ was assumed to be compact, $\alpha = d\tau$ has at least one zero. Hence, Proposition 4.16 cannot be applied in this case. However, if $b_1(\mathcal{B}) = 0$, we may choose a different vector field to split the first line from the universal covering. Indeed, if we choose the complete vector field $W := \zeta - 2(\tau \circ \pi)\xi$ on $\mathcal{M}$ instead of ζ, we can use the flow of its lift to the universal cover to split a line from $\widetilde{\mathcal{M}}$ just as within the proof of Proposition 4.16 but with the difference that now $[W, \mathrm{e}_\alpha] = 0$. We obtain the following.

Proposition 4.21. *Let $(\mathcal{M}^{(n+2)}, g)$ be a Lorentzian manifold of type (Ψ, A, η, f) over $(\mathcal{N}, h)$ as in Proposition 4.16 but with $b_1(\mathcal{B}) = 0$. Then the universal cover $(\widetilde{\mathcal{M}}, \widetilde{g})$ is isometric to a manifold*

$$(\mathbb{R}^2 \times \mathcal{S},\ \Xi_{(u,v,p)} = 2dudv + \kappa(u,p)du^2 + \Theta_p) \tag{4.29}$$

with the notations as in Proposition 4.16.

Proof. Let $\alpha = d\tau$ and define $W := \zeta - 2(\tau \circ \pi)\xi$. Then $\pi^*\eta(W) = 1$ and the same methods as in the proof of Proposition 4.16 apply. Namely, by taking the flow $\{\varphi_u^W\}_{u\in\mathbb{R}}$ of $\widetilde{W}$ and $\{\varphi_v^\xi\}_{v\in\mathbb{R}}$ of $-\widetilde{\xi}$ we can separate a line from $\widetilde{\mathcal{M}}$ twice by Proposition 1.44:

$$\Phi : \mathbb{R} \times \mathbb{R} \times S \overset{\varphi_v^\xi}{\simeq} \mathbb{R} \times L^\perp \overset{\varphi_u^W}{\simeq} \widetilde{\mathcal{M}}.$$

Again we will use, locally, as a basis of $T\mathcal{B}$ of $h_\mathcal{B}$-orthonormal vector fields $E_\alpha, E_1, \ldots, E_n$ with $E_\alpha := \frac{\alpha^\sharp}{||\alpha^\sharp||}$ and $E_2, \ldots, E_n \in \ker \alpha$ and follow the notations in the proof of Proposition 4.16. Then:

$$[W, \mathrm{e}_\alpha] = \mathrm{i}F^A(\zeta, \mathrm{e}_\alpha)\xi + 2d\tau(\mathrm{e}_\alpha)\xi = -2d\tau(E_\alpha)\xi + 2d\tau(E_\alpha)\xi = 0 \text{ and } [W, \mathrm{e}_i] = 0$$

for $i = 2, \ldots, n$. We obtain:

$$d\Phi(\partial_u) = \widetilde{W},\ d\Phi(\partial_v) = -\widetilde{\xi} \text{ and } d\Phi(\mathrm{e}_i) \in \Gamma(\widetilde{\ker \pi^*\eta}).$$

The assertion now follows, since by the former equations,

$$\Phi^*(\widetilde{iA}) = dv + 2(\widetilde{\tau \circ \pi} \circ \Phi) \cdot du.$$

Setting $\kappa(u, p) := (\widetilde{f} + 1 + 4\widetilde{\tau \circ \pi}) \circ \Phi(u, 0, p)$ completes the proof. □

Note that if not only $b_1(\mathcal{B}) = 0$ but even $\mathcal{B}$ is simply-connected, $\mathcal{M}$ is diffeomorphic to $\mathbb{T}^2 \times \mathcal{B}$ since in this case the circle bundle is trivial as $[\Psi] = 0$. However, this must in general not be the case. Therefore it seems to be worthwhile to mention that the same conclusion about the holonomy as in Theorem 4.20 also holds for the case when $b_1(\mathcal{B}) = 0$:

Corollary 4.22. *Let $(\mathcal{M}^{(n+2)}, g)$ be a Lorentzian manifold of type (Ψ, A, η, f) over $(\mathcal{N}, h)$ with the data chosen as in Proposition 4.21 with $f \in C^\infty(\mathcal{N})$ s.t.* $\mathrm{Hess}_B f|_\mathcal{B}$ *is non-degenerate in a point. Then the full holonomy of $(\mathcal{M}, g)$ is given as in Theorem 4.20.*

Complete Examples with Type 4 Holonomy

Obviously, we have so far just considered Lorentzian manifolds of type (Ψ, A, η, f) over $(\mathcal{N}, h)$ which are of type 1 or 2, cf. Theorem 1.7. By [Bez05, Proposition 6.2], they cannot be of type 3 since $R^{\nabla^\xi}(e_+, \xi) = 0$ if and only if $\xi(f) = 0$.[5] However, as we will see, for appropriate choices of the objects, we can obtain Lorentzian manifolds of type 4 which are complete (but non-compact). We do not know if the other existing examples [Gal06, Baz09, Lei06] for Lorentzian manifolds with holonomy of type 4 provide complete examples, too.

For the purpose of the construction we use a characterization of type 4 holonomy algebras contained in [Bez05, Proposition 6.3]. It is basically a consequence of the Holonomy Theorem of Ambrose and Singer, subsumed in equation (1.4), and a certain curvature decomposition, see [LG08, Theorem 3.7].

Proposition 4.23. *A Lorentzian manifold $(\mathcal{M}^{(n+2)}, g)$ of type (Ψ, A, η, f) over $(\mathcal{N}, h)$ has type 4 holonomy algebra $\mathfrak{hol}_x(\mathcal{M}^{(n+2)}, g)$ in $x \in \mathcal{M}$ if and only if there is a screen distribution $\mathbb{S}$ and a decomposition $\mathbb{S} = \mathbb{S}_1 \oplus \mathbb{S}_2$ such that the following holds true.*

(i) $R^{\nabla^{\mathbb{S}}}(X,Y)\Gamma(\mathbb{S}_1) \subset \Gamma(\mathbb{S}_1)$ *and* $R^{\nabla^{\mathbb{S}}}(X,Y)\Gamma(\mathbb{S}_2) = 0$ *for all* $X, Y \in \Gamma(T\mathcal{M})$.

(ii) There exists a section $\varphi \in \Gamma(\mathrm{Hom}(\mathfrak{so}(\mathbb{S}_1), \mathbb{S}_2))$ *s.t.*

a) $R^{\nabla^{\mathbb{S}}}(X,Y) \in \ker\varphi$ *for all* $X, Y \in \Gamma(\mathbb{S})$,

b) $\widehat{R}(e_+, \Gamma(\mathbb{S}_2))\Gamma(\mathbb{S}_2) = 0$ *and* $\widehat{R}(e_+, X)Y = g(\varphi(R^{\nabla^{\mathbb{S}}}(e_+, X)), Y)\xi$ *for all vector fields* $X \in \Gamma(\mathbb{S}_1)$ *and* $Y \in \Gamma(\mathbb{S}_2)$, *where* $\widehat{R} = R^g - R^{\nabla^{\mathbb{S}}}$.

c) For any $y \in \mathcal{M}$ *and* $\gamma : [0,1] \longrightarrow \mathcal{M}$ *with* $\gamma(0) = x$ *and* $\gamma(1) = y$ *it holds*

$$g_y(\varphi_y(R_y^{\nabla^{\mathbb{S}}}(e_+, X)), \mathcal{P}_\gamma^g(Y(x))) = g_x(\varphi_x(\mathrm{pr}_{\mathbb{S}_1} \circ \mathcal{P}_{\gamma^-}^g \circ R_y^{\nabla^{\mathbb{S}}}(e_+, X) \circ \mathcal{P}_\gamma^g \circ \mathrm{pr}_{\mathbb{S}_1}), Y(x))$$

for arbitrary $X \in \Gamma(\mathbb{S}_1)$ *and* $Y \in \Gamma(\mathbb{S}_2)$.

Applying this to a certain family of Lorentzian manifolds of type (Ψ, A, η, f) over the manifold $\mathcal{N} = \mathbb{R}^m \times \mathbb{T}^k$ gives us the following.

Proposition 4.24. *Let $(\mathcal{M}^{(n+2)}, g)$ be a Lorentzian manifold of type (Ψ, A, η, f) over $(\mathcal{N}, h)$, where we choose $\mathcal{N} = \mathcal{B} \times \mathbb{S}^1$ with $\mathcal{B} = \mathbb{R}^m \times \mathbb{T}^k$, $\frac{k(k-1)}{2} \geq m > 0$ and $k \geq 2$. Denote by $\eta = du$ the coordinate 1-form on $\mathbb{S}^1$ and fix a global trivialization of $T\mathcal{B}$ by $\partial_1, \ldots, \partial_m, E_1, \ldots, E_k$, where $E_1, \ldots, E_k$ is an orthonormal frame w.r.t. the flat metric θ on $\mathbb{T}^k$. Furthermore, choose*

- $0 \neq [\Psi] \in H^2_{\mathrm{dR}}(\mathbb{T}^k) \cap H^2(\mathbb{T}^k, \mathbb{Z})$ *for a non-harmonic* Ψ *and* $[\Psi(x), \Psi(y)]_{\mathfrak{so}(k)} = 0$ *for all* $x, y \in \mathbb{T}^k$, *where* $\Psi(x)$ *is understood as an element of* $\mathfrak{so}(k)$ *w.r.t. the basis* $E_1, \ldots, E_k$;
- *smooth non-zero functions* $\varphi_i : \mathbb{R} \longrightarrow \mathbb{R}^*$ *with* $\varphi_i(0) = 1$ *for* $i = 1, \ldots, m$;
- $h = h_{\mathcal{B}} \oplus du^2$ *with* $h_{\mathcal{B}} = \sum_{i=1}^m \varphi_i^2 dx_i^2 \oplus \theta$,
- $f := \widehat{f} \circ \pi$ *for* $\widehat{f} \in C^\infty(\mathcal{B})$ *with* $\widehat{f}(y_1, \ldots, y_m, x) := -2\sum_{(i,j)\in\Lambda} \Psi_{ij}(x)\Phi_{\lambda_i^j}(y_{\lambda_i^j})$, *where* Φ_i *is the antiderivative of* φ_i *with* $\Phi_i(0) = C_i \in \mathbb{R}$ *and whereby* $\lambda_i^j := \frac{(j-2)(j-1)}{2} + i$ *and* $\Lambda := \{(i,j) \in \{1, \ldots, k\}^2 \mid i < j,\ \lambda_i^j \leq m\}$.

Then $(\mathcal{M}^{(n+2)}, g)$ is a Lorentzian manifold with holonomy of type 4 and Abelian orthogonal part $\mathfrak{g} \subset \mathfrak{so}(k)$, where $\dim\mathfrak{g} = \left\lfloor \frac{\operatorname{rank}\psi}{2} \right\rfloor$.

5 With ∇^ξ we denote the connection $\nabla^\xi : \Gamma(\mathbb{L}) \longrightarrow \Gamma(T^*\mathcal{M} \otimes \mathbb{L})$ with $\mathbb{L} = \mathbb{R}\xi$ induced by ∇^g.

Proof. By the construction of $\mathcal{M}$, we have that $\mathcal{M} = \mathbb{R}^m \times \mathcal{M}' \times \mathbb{S}^1$ for the $\mathbb{S}^1$-bundle $\pi : \mathcal{M}' \longrightarrow \mathbb{T}^k$ with $c_1(\mathcal{M}') = [\Psi]$. Let $\mathbb{S}$ be the screen distribution corresponding to the choice of the transversal vector field $Z \in \Gamma(T\mathcal{M})$ defined in (4.3) and $\nabla^{\mathbb{S}}$ denote the induced connection from ∇^g defined in (1.16) on page 19. By the choice of $\mathbb{S}$ we have $\mathbb{S} \simeq T\mathcal{B}^* = \operatorname{span}\{S_1^*, \ldots, S_n^*\}$ *globally*, where we set $S_i := \varphi_i^{-1}\partial_i$ and $S_j := E_j$ for $i = 1, \ldots, m$, $j = 1, \ldots, k$. Hence we have a splitting $\mathbb{S} = \mathbb{S}_2 \oplus \mathbb{S}_1$ with $\mathbb{S}_1 = (T\mathbb{T}^k)^*$ and $\mathbb{S}_2 = \mathbb{R}^m$.

To this end we fix the point $x = (0, p, u) \in \mathcal{M}$ for arbitrary $p \in \mathcal{M}'$ and $u \in \mathbb{S}^1$. Since the holonomy algebras in different points of the manifold are isomorphic, it suffices to prove that $\mathfrak{hol}_x(\mathcal{M}^{(n+2)}, g)$ is of type 4. Computing $R^{\nabla^{\mathbb{S}}}$ using Lemma 4.7 we see that

$$R^{\nabla^{\mathbb{S}}}(\cdot,\cdot)S = (\nabla^h_{\cdot}\psi)(S) \wedge \pi^*\eta \tag{4.30}$$

for all $S \in \Gamma(\mathbb{S})$ and thus

$$R^{\nabla^{\mathbb{S}}}(X,Y)\Gamma(\mathbb{S}_1) \subset \Gamma(\mathbb{S}_1), R^{\nabla^{\mathbb{S}}}(\Gamma(\mathbb{S}),\Gamma(\mathbb{S})) = 0, R^{\nabla^g}(X,Y)\Gamma(\mathbb{S}_2) = R^{\nabla^{\mathbb{S}}}(X,Y)\Gamma(\mathbb{S}_2) = 0$$

for all $X, Y \in \Gamma(T\mathcal{M})$. Therefore, in Proposition 4.23 the properties (i), (ii.a) and the first equation in (ii.b) are satisfied.

We are left to choose a section $\varphi \in \Gamma(\operatorname{Hom}(\mathfrak{so}(\mathbb{S}_1), \mathbb{R}^m))$ for which (ii.b) and (ii.c) in Proposition 4.23 hold. For every $y = (y_1, \ldots, y_m, q, v) \in \mathcal{M}$,

$$\varphi_y : \mathfrak{so}((\mathbb{S}_1)_y) \ni A_y \longmapsto \sum_{(i,j)\in\Lambda} A_{ij}(y)\varphi_{\lambda_i^j}(y_{\lambda_i^j})\partial_{\lambda_i^j} \in \mathbb{R}^m \tag{4.31}$$

defines a surjective linear map[6]. To prove that Proposition 4.23 (ii.b) is satisfied we compute

$$\widehat{R}(e_+, X)Y = \tfrac{1}{2}(\operatorname{Hess}_g f)(X,Y)\xi$$

for all $X \in \Gamma(\mathbb{S}_1)$ and $Y \in \Gamma(\mathbb{S}_2)$. Moreover, we obtain for the Hessian of f

$$(\operatorname{Hess}_g f)_y(X, \partial_\ell) = X(\partial_\ell(f))(y) = -2X(\Psi_{i_0 j_0})\varphi_\ell(y_\ell) = -2g_y(\varphi(\nabla^h_{d\pi(X)}\psi), \partial_\ell) \tag{4.32}$$

for $\lambda_{i_0}^{j_0} = \ell$ and all $X \in \Gamma(\mathbb{S}_1)$ since $\varphi_\ell = \partial_\ell(\Phi_\ell)$. Therefore,

$$\widehat{R}(e_+, X)Y = g(\varphi(R^{\nabla^{\mathbb{S}}}(e_+, X)), Y)\xi$$

for all $X \in \Gamma(\mathbb{S}_1)$ and $Y \in \Gamma(\mathbb{S}_2)$ by (4.30) and (4.32) which proves Proposition 4.23 (ii.b).

Hence it remains to show that Proposition 4.23 (ii.c) holds. Let $\gamma : [0,1] \longrightarrow \mathcal{M}$ be a path with $\gamma(0) = x$ and $\gamma(1) = y$. Since $\mathcal{M} = \mathbb{R}^m \times \mathcal{M}' \times \mathbb{S}^1$ we can write the curve γ as $\gamma(t) = (\sigma(t), \overline{\delta}, e^{iu(t)})$ for appropriate $\sigma : [0,1] \longrightarrow \mathbb{R}^m$, $\overline{\delta} : [0,1] \longrightarrow \mathcal{M}'$ and $\delta := \pi \circ \overline{\delta}$. Then one computes for the parallel transport of any $E \in \{\partial_1, \ldots, \partial_m, E_1^*, \ldots, E_k^*\}$ along γ that

$$\mathcal{P}^g_\gamma(E) = C_V \cdot \xi + V^*(1),$$

6 For the purpose of clarifying this definition, we point out that the presented homomorphism φ is nothing but the restriction of the canonical isomorphism $\mathfrak{so}(k) \cong \mathbb{R}^{k(k-1)/2}$ given by the function $(a_{ij}) \mapsto (a_{12}, a_{13}, \ldots, a_{1k}, a_{23}, \ldots, a_{(k-1)k})$ to the first m entries and weighted by the non-vanishing functions φ_i, $i = 1, \ldots, m$.

where $C_V \in \mathbb{R}$ depends on $V \in \Gamma(\delta^* T\mathbb{T}^k \oplus \sigma^* \mathbb{R}^m)$ which is the solution to the ODE

$$\nabla^h_{\dot{\delta}} V = -\dot{u} \cdot \psi(V) \tag{4.33}$$

with initial value $V(0) = d\pi(E)$. When $E = \partial_i$, then $\psi(E) = 0$ and we obtain the solution

$$V(t) = \mathcal{P}^h_{\delta|_{[0,t]}}(\partial_i(\delta(0))) = \frac{1}{\varphi_i(\delta_i(t))}\partial_i(\delta(t)). \tag{4.34}$$

Hence, to solve (4.33) we can write down (4.33) as matrix equation of $(k \times k)$-matrices

$$\dot{\Omega}(t) = A(t) \cdot \Omega(t),\ \Omega(0) = \mathbb{I}_k \tag{4.35}$$

for $A(t) := -\dot{u}(t) \cdot \psi(\delta(t))$, where $\mathbb{I}_k$ is the identity and ψ is interpreted as an element of $\mathfrak{so}(k)$. We conclude that $\Omega(t) \in \mathrm{SO}(k)$ since $A(t) \in \mathfrak{so}(k)$. We obtain

$$\mathrm{pr}_{\mathrm{S}_1} \circ \mathcal{P}^g_\gamma \circ \mathrm{pr}_{\mathrm{S}_1} = \Omega_s(1) \in \mathrm{SO}(k) \tag{4.36}$$

where Ω_s is the solution to (4.35). By [Mag54], the solution Ω_s can explicitly written down as

$$\Omega_s(t) = \exp\left(\int_0^t A(\tau)\mathrm{d}\tau\right) \tag{4.37}$$

since $[\Psi(\delta(\tau_1)), \Psi(\delta(\tau_2))]_{\mathfrak{so}(k)} = 0$ for all $\tau_1, \tau_2 \in [0,1]$ implying $[A(\tau_1), A(\tau_2)]_{\mathfrak{so}(k)} = 0$. Equation (4.37) in turn implies that

$$\Omega_s(t_1)\Omega_s(t_2) = \Omega_s(t_2)\Omega_s(t_1) \tag{4.38}$$

for $t_1, t_2 \in [0,1]$, which, by setting $t_2 = 1$ and differentiating in $t_1 = 1$, yields

$$\Psi(\delta(1))\Omega_s(1) = \Omega_s(1)\Psi(\delta(1)). \tag{4.39}$$

We are now in the position to prove Proposition 4.23 (ii.c). Consider the left hand side of the equation occurring in Proposition 4.23 (ii.c) for $X \in \Gamma(\mathrm{S}_1)$, $Y = \partial_i$ and $y = (y_1, \dots, y_m, q, v) \in \mathcal{M}$. We compute:

$$\begin{aligned} g_y(\varphi_y(R^{\nabla^{\mathrm{S}}}_y(\mathrm{e}_+, X)), \mathcal{P}^g_\gamma(\partial_i(x))) &= \varphi_i(y_i)^{-1} \cdot g_y(\varphi_y(R^{\nabla^{\mathrm{S}}}_y(\mathrm{e}_+, X)), \partial_i(y)) \\ &= -\varphi_i(y_i)^{-1} \cdot X(\Psi_{i_0 j_0})(q)\varphi_i(y_i) \\ &= -X(\Psi_{i_0 j_0})(q) \end{aligned} \tag{4.40}$$

where $i_0, j_0 \in \{1, \dots, k\}$ such that $\lambda^{j_0}_{i_0} = i$. To compute the right hand side, define

$$B_{ij}(x) := g_p(\mathrm{pr}_{\mathrm{S}_1} \circ \mathcal{P}^g_{\gamma^-} \circ R^{\nabla^{\mathrm{S}}}_y(\mathrm{e}_+, X) \circ \mathcal{P}^g_\gamma(E^*_i(x)), E^*_j(x)),$$

such that it turns into

$$g_x(\varphi_x(\mathrm{pr}_{\mathrm{S}_1} \circ \mathcal{P}^g_{\gamma^-} \circ R^{\nabla^{\mathrm{S}}}_y(\mathrm{e}_+, X) \circ \mathcal{P}^g_\gamma \circ \mathrm{pr}_{\mathrm{S}_1}), \partial_i(x)) = g_x(\varphi_x(B(x)), \partial_i(x)).$$

We compute

$$B_{ij}(x) \quad = \quad g_x(\mathcal{P}^g_{\gamma^-} \circ R^{\nabla^{\mathrm{S}}}_y(\mathrm{e}_+, X) \circ \mathcal{P}^g_\gamma(E^*_i(x)), E^*_j(x))$$

$$
\begin{aligned}
&= \quad g_x(R_y^{\nabla^S}(e_+, X) \circ \mathcal{P}_\gamma^g(E_i^*(x)), \mathcal{P}_\gamma^g(E_j^*(x))) \\
&= \quad -g_y((\nabla_X^g \pi^*\Psi)(y) \circ \Omega_s(1)E_i^*(y), \Omega_s(1)E_j^*(y)) \\
&\overset{(4.39)}{=} \quad -g_y(\Omega_s(1) \circ (\nabla_X^g \pi^*\Psi)(y)(E_i^*(y)), \Omega_s(1)E_j^*(y)) \\
&= \quad -g_y((\nabla_X^g \pi^*\Psi)(y)(E_i^*(y)), E_j^*(y)) \\
&= \quad -(\nabla_X^g \pi^*\Psi)_{ij}(y) \\
&= \quad -X(\Psi_{ij})(q). \qquad (4.41)
\end{aligned}
$$

Using this, we infer

$$g_x(\varphi_x(B(x)), \partial_i(x)) \overset{(4.31)}{=} B_{i_0 j_0}(x)\varphi_i(0) \overset{(4.41)}{=} -X(\Psi_{i_0 j_0})(q).$$

Taking into account (4.40) this shows Proposition 4.23 (ii.c) and completes the proof of the proposition. ☐

Remark 4.25. *To our knowledge, up to now no compact examples of Lorentzian manifolds with holonomy algebra of type 4 do exist. Unfortunately we do not know, how to replace the $\mathbb{R}^m$ factor in $\mathcal{M}$ by some compact manifold of dimension m (e.g. the torus). The simplest idea is to try to choose periodic functions φ_i such that their antiderivative is a periodic function. But since φ_i needs to be non-vanishing (i.e. either positive or negative), this is impossible.*

Under additional assumptions we get completeness of the latter manifolds producing examples for geodesically complete Lorentzian manifolds with holonomy of type 4.

Lemma 4.26. *If the functions φ_i, $i = 1, \ldots, m$, and $[\Psi] \in H^2_{\mathrm{dR}}(\mathbb{T}^k) \cap H^2(\mathbb{T}^k, \mathbb{Z})$ can be chosen, such that $(\mathbb{R}^m, \sum_{i=1}^m \varphi_i^2 dx_i^2)$ is complete and for each $u \in \mathbb{R}$ the solutions $s \mapsto \delta(s) \in \mathcal{B}$ to the equation*

$$\frac{\nabla^h \dot\delta}{ds}(s) = \frac{u^2}{2} \operatorname{grad}_h \widehat{f}(\delta(s)) - u \cdot \psi(\dot\delta) \qquad (4.42)$$

are defined on the whole real line, then the Lorentzian manifold of type (Ψ, A, η, f) over $(\mathcal{N}, h)$ in Proposition 4.24 is complete.

Proof. Let $\gamma : t \mapsto \gamma(t) \in \mathcal{M}$ be a curve with $\gamma(t) = (\alpha(t), e^{\mathrm{i}u(t)})$, where $\alpha : t \mapsto \alpha(t)$ is a curve in $\mathbb{R}^m \times \mathcal{M}'$ and define $\delta := \pi \circ \alpha$. Note that, by abuse of notation, we write for π the projection $\pi : \mathbb{R}^m \times \mathcal{M}' \longrightarrow \mathbb{R}^m \times \mathbb{T}^k$ to make notation short. Indeed, π restricted to $\mathbb{R}^m$ is just the identity. We get

$$\dot\gamma(t) = \dot u(t)\partial_u + \dot\alpha(t) = \dot u(t)\partial_u + v(t)\xi(t) + dr_{\rho(t)}(\dot\delta^*(t))$$

with δ^* denoting the horizontal lift of δ with $\delta^*(0) = \alpha(0)$ and $r_u : \mathcal{M}' \longrightarrow \mathcal{M}'$ the right action of $u \in \mathbb{S}^1$ on $\mathcal{M}'$, while $\rho : \mathbb{R} \to \mathbb{S}^1$ is defined through $r_{\rho(t)}(\delta^*(t)) := \alpha(t)$. This yields

$$\tfrac{\nabla^g \dot\gamma}{dt}(t) = \ddot u(t)\partial_u + (\dot v(t) - \dot u(t) d\widehat{f}(\dot\delta))\xi(t) + \dot u(t)(\psi(\dot\delta) - \tfrac{1}{2}\dot u(t) \operatorname{grad}_h \widehat{f}) + \tfrac{\nabla^h \dot\delta}{dt}(t). \quad (4.43)$$

Let $x = (y, p, e^{\mathrm{i}u_0}) \in \mathcal{M}$ and $v \in T_x\mathcal{M}$ be arbitrary with $v = u_1 \cdot \partial_u + \lambda \cdot \xi(x) + w$ where $w \in T_x\mathcal{B}^* \cong \mathbb{R}^m \oplus \mathcal{H}_p\mathcal{M}'$.[7] To prove completeness, we have to provide a geodesic

7 For any principal bundle $\mathcal{P} \longrightarrow \mathcal{B}$ we denote by $\mathcal{HP}$ its horizontal bundle.

$\gamma : \mathbb{R} \longrightarrow \mathcal{M}$ with $\gamma(0) = x$ and $\dot{\gamma}(0) = v$ defined on the whole real line. Let $\delta : \mathbb{R} \longrightarrow \mathcal{B}$ be a solution to (4.42) for $u = u_1$ with $\delta(0) = \pi(y, p)$ and $\dot{\delta}(0) = d\pi(w)$. By (4.43), for each geodesic γ we have $u(t) = u_1 t + u_0$. Therefore, $\tau(t) := \dot{u}(t) d\widehat{f}(\dot{\delta})$ is defined on the whole $\mathbb{R}$ and we define by $\mathcal{T} : \mathbb{R} \longrightarrow \mathbb{R}$ its antiderivative with $\mathcal{T}(0) = \lambda$. If $T : \mathbb{R} \longrightarrow \mathbb{R}$ is the antiderivative of $\mathcal{T}$ with $T(0) = 0$, then we define by

$$\alpha(t) := r_{\rho(t)}(\delta^*(t))$$

for $\rho(t) := e^{\mathrm{i}T(t)}$ a curve in $\mathbb{R}^m \times \mathcal{M}'$ with δ^* denoting the horizontal lift of δ with $\delta^*(0) = (y, p)$. We do now claim that

$$\gamma(t) := (\alpha(t), e^{\mathrm{i}(u_1 t + u_0)})$$

is the required geodesic with $\gamma(0) = x$ and $\dot{\gamma}(0) = v$. To see this, first note that

$$\gamma(0) = (\alpha(0), e^{\mathrm{i}u_0}) = (\delta^*(0), e^{\mathrm{i}u_0}) = x.$$

In order to verify that γ is a g-geodesic, recall (4.43) and the formula

$$\frac{d}{dt}\alpha(t) = dr_{\rho(t)}(\tfrac{d}{dt}\delta^*(t)) + X(\alpha(t)),$$

where $X \in \Gamma(T\mathcal{M}')$ is the fundamental vector field corresponding to $dL_{\rho(t)^{-1}}(\dot{\rho}(t)) \in \mathrm{i}\mathbb{R}$ with $L_u : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ denoting the left-multiplication by u in $\mathbb{S}^1$. In fact,

$$X(\alpha(t)) = \mathcal{T}(t) \cdot \xi(t),$$

while $\pi \circ \alpha = \delta$. Hence $\dot{v}(t) = \tau(t) = \dot{u}(t) d\widehat{f}(\dot{\delta})$ and since δ satisfies (4.42), γ is a g-geodesic with $\dot{\gamma}(0) = v$. □

The following result provides an example for the existence of the required functions φ_i and the 2-form Ψ such that (4.42) in Lemma 4.26 is satisfied.

Proposition 4.27. *Let $\xi^1, \dots, \xi^k$ be the canonical coframe on $\mathbb{T}^k$. Moreover choose $\varphi_i \equiv 1$, $i = 1, \dots, m$, and for $l = \left\lfloor \frac{k}{2} \right\rfloor$ define*

$$\Psi(x_1, \dots, x_k) := \sum_{i=1}^{l} \chi_i(x_{2i-1}, x_{2i}) \xi^{2i-1} \wedge \xi^{2i} \in \mathfrak{so}(2l) \subset \mathfrak{so}(k)$$

for periodic functions $\chi_i : \mathbb{T}^2 \longrightarrow \mathbb{R}$, $i = 1, \dots, 2l$ such that $[\Psi] \in H^2_{\mathrm{dR}}(\mathbb{T}^k) \cap H^2(\mathbb{T}^k, \mathbb{Z})$.[8] *Then the Lorentzian manifolds of type (Ψ, A, η, f) over $(\mathcal{N}, h)$ provided in Proposition 4.24 are geodesically complete.*

8 The constructed Ψ is of the form $\Psi = \Psi_1 + \ldots + \Psi_l$ where each $\Psi_i \in \Omega^2(\mathbb{T}^2)$ is simply a 2-form on $\mathbb{T}^2$ not depending on the other coordinates. Hence we obtain $[\Psi] \in H^2(\mathbb{T}^k, \mathbb{Z})$ if and only if the integral over the fundamental class $[\mathbb{T}^2]$ for each Ψ_i is an integer. For example one may choose the functions $\chi_i(x_i, x_{i+1}) := \sin(x_i)\sin(x_{i+1})$.

Proof. Let $\delta(s) = (\alpha_1(s), \dots, \alpha_m(s), \beta_1(s), \dots, \beta_k(s))$ be a path in $\mathcal{B} = \mathbb{R}^m \times \mathbb{T}^k$ and set $\alpha(s) = (\alpha_1(s), \dots, \alpha_m(s))$ and $\beta(s) = (\beta_1(s), \dots, \beta_k(s))$. Since $h_{\mathcal{B}} = h_1 \oplus h_2$ with $h_1 = \sum_{i=1}^m \varphi_i^2 dx_i^2$ and $h_2 := \theta$, equation (4.42) becomes

$$\left.\begin{array}{rcl} \ddot{\alpha}(s) & = & \frac{u^2}{2}\operatorname{grad}_{h_1}\widehat{f}(\delta(s)), \\ \frac{\nabla^{h_2}\dot{\beta}}{ds}(s) & = & \frac{u^2}{2}\operatorname{grad}_{h_2}\widehat{f}(\delta(s)) - u\psi(\dot{\beta}(s)). \end{array}\right\} \tag{4.44}$$

Taking into account the definition of $\widehat{f} \in C^\infty(\mathcal{B})$ in Proposition 4.24 and $\Phi_i(x) = x + C_i$ for $i = 1, \dots, m$, equation (4.44) turns into

$$\left.\begin{array}{rcl} \ddot{\alpha}_a(s) & = & -u^2\Psi_{i_0j_0}(\beta(s)),\ a = 1, \dots, m, \\ \frac{\nabla^{h_2}\dot{\beta}}{ds}(s) & = & -u^2\sum_{b=1}^k \sum_{(i,j)\in\Lambda}\{E_b(\Psi_{ij})(\beta(s))(\alpha_{\lambda_i^j}(s) + C_{\lambda_i^j})\}E_b - u\psi(\dot{\beta}(s)) \end{array}\right\} \tag{4.45}$$

where $i_0, j_0 \in \{1, \dots, k\}$ such that $\lambda_{i_0}^{j_0} = a$. By integrating the first equation of (4.45) twice and substituting this into the second equation we obtain equivalently:

$$\frac{\nabla^{h_2}\dot{\beta}}{ds}(s) = -u^4\sum_{b=1}^k \sum_{(i,j)\in\Lambda}\left\{E_b(\Psi_{ij})(\beta(s))\left(C_{\lambda_i^j} - \int_0^s\int_0^t \Psi_{ij}(\beta(\tau))\mathrm{d}\tau\mathrm{d}t\right)\right\}E_b - u\psi(\dot{\beta}(s)) \tag{4.46}$$

By lifting this equation to $\mathbb{R}^k$, we obtain a second order non-linear differential equation of the form $y''(s) = F(s, y, y') := A(s, y(s)) + B(y(s))y'(s)$. Since the partial derivatives of $A : [a, b] \times \mathbb{R}^k \longrightarrow \mathbb{R}$ and $B : \mathbb{R}^k \longrightarrow \mathbb{R}$ are bounded, $F : \mathbb{R}^{2k} \longrightarrow \mathbb{R}^k$ is globally Lipschitz continuous and (4.46) exhibits a global solution $\widetilde{\beta} : \mathbb{R} \longrightarrow \mathbb{R}^k$. By defining $\beta := p \circ \widetilde{\beta}$ for p being the canonical projection $p : \mathbb{R}^k \longrightarrow \mathbb{T}^k$ then yields the global solution on the torus. $\square$

Combining Proposition 4.24 and Proposition 4.27 we finally obtain the following result.

Theorem 4.28. *For each Abelian Riemannian holonomy algebra $\mathfrak{g} \subset \mathfrak{so}(k)$ there exists a complete indecomposable but non-irreducible Lorentzian manifold with holonomy of type 4 possessing $\mathfrak{g}$ as orthogonal part.*

BIBLIOGRAPHY

[Aub98] T. Aubin, *Some Nonlinear Problems in Riemannian Geometry*, 1st ed., Springer Verlag, 1998.

[Bau02] H. Baum, *Conformal Killing spinors and special geometric structures in Lorentzian geometry – a survey*, Proceedings of the Workshop on Special Geometric Structures in String Theory, Bonn, September 2001, Proceedings archive of the EMS Electronic Library of Mathematics, 2002, http://www.univie.ac.at/EMIS/proceedings/.

[Bau12] ———, *Holonomy Groups of Lorentzian Manifolds: A Status Report*, Global Differential Geometry (Christian Bär, Joachim Lohkamp, and Matthias Schwarz, eds.), Springer Proceedings in Mathematics, vol. 17, Springer Berlin Heidelberg, 2012, 10.1007/978-3-642-22842-1_7, pp. 163–200.

[Baz09] Ya. V. Bazaĭkin, *Globally hyperbolic lorentzian spaces with special holonomy groups*, Siberian Mathematical Journal **50** (2009), no. 4, 567–579.

[BBI93] L. Bérard-Bergery and A. Ikemakhen, *On the holonomy of Lorentzian manifolds*, Differential Geometry: Geometry in Mathematical Physics and Related Topics (Los Angeles, CA, 1990), Proc. Sympos. Pure Math., vol. 54, Amer. Math. Soc., Providence, RI, 1993, pp. 27–40. MR MR1216527

[Ber55] M. Berger, *Sur les groupes d'holonomie homogène des variétés riemanniennes*, Bulletin de la Société Mathématique de France **83** (1955), 279–330.

[Ber57] ———, *Les espaces symétriques noncompacts*, Annales scientifiques de l'École Normale Supérieure **74** (1957), no. 2, 85–177.

[Bes87] A. L. Besse, *Einstein manifolds*, 1st ed., Springer, 1987.

[Bez05] N. Bezvitnaya, *Lightlike foliations on Lorentzian manifolds with weakly irreducible holonomy algebra*, preprint, arXiv:math.DG/0506101, 2005.

[Bla09] M. Blau, *Plane Wave Geometry and Quantum Physics*, Quantum Field Theory, Springer, 2009, pp. 197–216.

[BLL14] H. Baum, K. Lärz, and T. Leistner, *On the full holonomy group of Lorentzian manifolds*, Mathematische Zeitschrift **277** (2014), no. 3-4, 797–828 (English).

[BO03] M. Blau and M. O'Loughlin, *Homogeneous plane waves*, Nuclear Physics B **654** (2003), 135–176.

[Bra69] R. G. Brasher, *The Homology Sequence of the Double Covering; Betti Numbers and Duality*, Proceedings of the American Mathematical Society **23** (1969), no. 3, 714–717.

[Bri25] H. W. Brinkmann, *Einstein spaces which are mapped conformally on each other*, Mathematische Annalen **94** (1925), no. 1, 119–145.

[Car81] Y. Carrière, *Flots riemanniens et feuilletages géodésibles de codimension un*, Ph.D. thesis, Université des Sciences et Techniques, 1981.

[Car84] ———, *Flots riemanniens*, Astérisque **116** (1984), 31–52.

[Car89] ———, *Autour de la conjecture de L. Markus sur les variétés affines*, Invent. Math. **95** (1989), no. 3, 615–628. MR 979369 (89m:53116)

[CC00] A. Candel and L. Conlon, *Foliations I*, Graduate Studies in Mathematics, no. 23, American Mathematical Society, 2000.

[CFS03] A. M. Candela, J. L. Flores, and M. Sánchez, *On general plane fronted waves. Geodesics*, Gen. Relativity Gravitation **35** (2003), no. 4, 631–649. MR MR1971289 (2004c:58026)

[CN85] C. Camacho and A.L. Neto, *Geometric theory of foliations*, Birkhäuser Boston, 1985.

[Con74] L. Conlon, *Transversally parallelizable foliations of codimension two*, Transactions of the American Mathematical Society **194** (1974), 79–102.

[Con08] L. Conlon, *Differentiable manifolds*, 2nd ed., Birkhäuser, 2008.

[CRS12] A. M. Candela, A. Romero, and M. Sánchez, *Remarks on the completeness of plane waves and the trajectories of accelerated particles in Riemannian manifolds*, Proceedings of the International Meeting on Differential Geometry, Univ. of Córdoba, 2012, arXiv:1304.4818, pp. 15–17.

[CRS13] ———, *Completeness of the trajectories of particles coupled to a general force field*, Archive for Rational Mechanics and Analysis **208** (2013), no. 1, 255–274.

[CW70] M. Cahen and N. Wallach, *Lorentzian symmetric spaces*, Bulletin of the American Mathematical Society **76** (1970), no. 3, 585–591.

[Dom98] D. Domínguez, *Finiteness and tenseness theorems for Riemannian foliations*, American journal of Mathematics **120** (1998), no. 6, 1237–1276.

[DR52] G. De Rham, *Sur la reductibilite d'un espace de Riemann*, Commentarii Mathematici Helvetici **26** (1952), no. 1, 328–344.

[DR07] A. Derdziński and W. Roter, *Global properties of indefinite metrics with parallel Weyl tensor*, Pure and Applied Differential Geometry - PADGE 2007 (2007), 63–72.

[DR08] ———, *On compact manifolds admitting indefinite metrics with parallel Weyl tensor*, Journal of Geometry and Physics **58** (2008), no. 9, 1137–1147, arXiv:math.DG/0705.1106v1.

[DR09] ———, *The local structure of conformally symmetric manifolds*, Bulletin of the Belgian Mathematical Society - Simon Stevin **16** (2009), no. 1, 117–128.

[DR10] ———, *Compact pseudo-Riemannian manifolds with parallel Weyl tensor*, Annals of Global Analysis and Geometry **37** (2010), no. 1, 73–90, arXiv:math.DG/0702491v1.

[DSO01] A. J. Di Scala and C. Olmos, *The geometry of homogeneous submanifolds of hyperbolic space*, Mathematische Zeitschrift **237** (2001), 199–209, 10.1007/PL00004860.

[DZ10] S. Dumitrescu and A. Zeghib, *Géométries lorentziennes de dimension 3: classification et complétude*, Geom. Dedicata **149** (2010), 243–273. MR 2737692 (2012c:53105)

[EK62] J. Ehlers and W. Kundt, *Exact solutions of the gravitational field equations*, Gravitation: An introduction to current research, Wiley, New York, 1962, pp. 49–101. MR 0143624 (26 #1177)

[FQ90] M. H. Freedman and F. Quinn, *Topology of 4-Manifolds*, Princeton Legacy Library, Princeton University Press, 1990.

[FS06] J. L. Flores and M. Sánchez, *On the geometry of pp-wave type spacetimes*, Analytical and numerical approaches to mathematical relativity, Lecture Notes in Phys., vol. 692, Springer, Berlin, 2006, pp. 79–98. MR 2222548 (2007b:58025)

[FW75] A. E. Fischer and J. A. Wolf, *The structure of compact Ricci-flat Riemannian manifolds*, J. Differ. Geom. **10** (1975), 277–288 (English).

[Gal06] A. S. Galaev, *Metrics that realize all Lorentzian holonomy algebras*, Int. J. Geom. Methods Mod. Phys. **3** (2006), no. 5-6, 1025–1045. MR MR2264404

[GG09] D. Gonçalves and J. Guaschi, *The lower central and derived series of the braid groups of the sphere*, Transactions of the American Mathematical Society **361** (2009), no. 7, 3375–3399.

[GP08] G. W. Gibbons and C. N. Pope, *Time-dependent multi-centre solutions from new metrics with holonomy Sim(n-2)*, Classical and Quantum Gravity **25** (2008), no. 12, 125015.

[Hat02] A. Hatcher, *Algebraic Topology*, Cambridge University Press, 2002.

[HR10] G. Habib and K. Richardson, *Modified differentials and basic cohomology for Riemannian foliations*, Journal of Geometric Analysis (2010), 1–29.

[Hul84] C. M. Hull, *Exact pp-wave solutions of 11-dimensional supergravity*, Physics Letters B **139** (1984), no. 1, 39–41.

[Joy96a] D. D. Joyce, *Compact 8-manifolds with holonomy Spin(7)*, Inventiones mathematicae **123** (1996), no. 3, 507–552.

[Joy96b] ———, *Compact Riemannian 7-manifolds with holonomy G2. I, II*, J. Diff. Geom **43** (1996), no. 2, 291–375.

[Kam93] Y. Kamishima, *Completeness of Lorentz manifolds of constant curvature admitting Killing vector fields*, J. Differential Geom. **37** (1993), no. 3, 569–601. MR 1217161 (94f:53116)

[Kat99] I. Kath, *Killing Spinors on Pseudo-Riemannian Manifolds*, Habilitationsschrift, Humboldt-Universität zu Berlin, 1999.

[Kli96] B. Klingler, *Complétude des variétés lorentziennes à courbure constante*, Math. Ann. **306** (1996), no. 2, 353–370. MR 1411352 (97g:53082)

[KT87] F. W. Kamber and P. Tondeur, *De rham-hodge theory for riemannian foliations*, Mathematische Annalen **277** (1987), no. 3, 415–431.

[Lär11] K. Lärz, *Global Aspects of Holonomy in Pseudo-Riemannian Geometry*, Ph.D. thesis, Humboldt-Universität zu Berlin, 2011.

[Lei02] T. Leistner, *Lorentzian manifolds with special holonomy and parallel spinors*, Rend. Circ. Mat. Palermo (2) Suppl. (2002), no. 69, 131–159. MR MR1972431 (2004c:53065)

[Lei06] ———, *Screen bundles of Lorentzian manifolds and some generalisations of pp-waves*, J. Geom. Phys. **56** (2006), no. 10, 2117–2134. MR MR2241741

[Lei07] ———, *On the classification of Lorentzian holonomy groups*, J. Differential Geom. **76** (2007), no. 3, 423–484. MR MR2331527

[LG08] T. Leistner and A. S. Galaev, *Holonomy groups of Lorentzian manifolds: classification, examples, and applications*, Recent developments in pseudo-Riemannian geometry, ESI Lect. Math. Phys., Eur. Math. Soc., Zürich, 2008, pp. 53–96. MR MR2436228 (2009j:53053)

[LG10] ———, *On the local structure of Lorentzian Einstein manifolds with parallel distribution of null lines*, Classical and Quantum Gravity **27** (2010), no. 22, 225003.

[LS13] T. Leistner and D. Schliebner, *Completeness of compact Lorentzian manifolds with abelian holonomy*, preprint, arXiv:math.DG/1306.0120, 2013.

[Mag54] W. Magnus, *On the exponential solution of differential equations for a linear operator*, Communications on pure and applied mathematics **VII** (1954), no. 4, 649–673.

[Mar73] J. Marsden, *On completeness of homogeneous pseudo-riemannian manifolds*, Indiana Univ. J. **22** (1972/73), 1065–1066. MR 0319128 (47 #7674)

[Mas00] A. Mason, *An application of stochastic flows to Riemannian foliations*, Houston Journal of Mathematics **26** (2000), no. 3, 481–515.

[Mil63] J. Milnor, *Morse theory*, Based on lecture notes by M. Spivak and R. Wells. Annals of Mathematics Studies, No. 51, Princeton University Press, Princeton, N.J., 1963. MR 0163331 (29 #634)

[MM03] I. Moerdijk and J. Mrcun, *Introduction to Foliations and Lie Groupoids*, Cambridge Studies in Advanced Mathematics, no. 91, Cambridge University Press, 2003.

[Moi52] E. E. Moise, *Affine Structures in 3-Manifolds: V. The triangulation Theorem and Hauptvermutung*, Annals of mathematics (1952), 96–114.

[MS85] P. Molino and V. Sergiescu, *Deux remarques sur les flots riemanniens*, manuscripta mathematica **51** (1985), no. 1, 145–161.

[Ols93] Z. Olszak, *On conformally recurrent manifolds, I: Special distributions*, Zesz. Nauk. Politech. Sl., Mat.-Fiz **68** (1993), 213–225.

[O'N83] B. O'Neill, *Semi-Riemannian geometry*, Academic Press, 1983.

[Pet06] P. Petersen, *Riemannian Geometry*, 2nd ed., Graduate Texts in Mathematics Series, Springer Verlag, 2006.

[Pri01] J. I. R. Prieto, *The Gysin sequence for Riemannian flows*, Contemporary Mathematics **288** (2001), 415–419.

[PZ13] P. Piccione and A. Zeghib, *Actions of discrete groups on stationary Lorentz manifolds*, Ergodic Theory and Dynamical Systems (2013), 1–34.

[RS93] A. Romero and M. Sánchez, *On the completeness of geodesics obtained as a limit*, Journal of mathematical physics **34** (1993), no. 8, 3768–3774.

[RS94a] ———, *On completeness of certain families of semi-Riemannian manifolds*, Geom. Dedicata **53** (1994), no. 1, 103–117. MR 1299888 (95g:53049)

[RS94b] ———, *On completeness of compact Lorentzian manifolds*, Geometry and topology of submanifolds, VI (Leuven, 1993/Brussels, 1993), World Sci. Publ., River Edge, NJ, 1994, pp. 171–182. MR 1315099 (96c:53106)

[RS95] ———, *Completeness of compact Lorentz manifolds admitting a timelike conformal Killing vector field*, Proc. Amer. Math. Soc. **123** (1995), no. 9, 2831–2833. MR 1257122 (95k:53075)

[RS96] ———, *An integral inequality on compact Lorentz manifolds, and its applications*, Bulletin of the London Mathematical Society **28** (1996), no. 5, 509–513.

[Sán97] M. Sánchez, *Structure of Lorentzian tori with a Killing vector field*, Transactions of the American Mathematical Society **349** (1997), no. 3, 1063–1080.

[Sán13] ———, *On the completeness of trajectories for some mechanical systems*, preprint, arXiv:math.DG/1303.6150, 2013.

[Sch61] J. F. Schell, *Classification of Four-Dimensional Riemannian Spaces*, Journal of Mathematical Physics **2** (1961), no. 2, 202–206.

[Sch12] D. Schliebner, *On the Full Holonomy of Lorentzian Manifolds with Parallel Weyl Tensor*, preprint, arXiv:math.DG/1204.5907, 2012.

[Sch13] ———, *On Lorentzian manifolds with highest first Betti number*, to appear in Annales de l'Institut Fourier, arXiv:math.DG/1311.6723, 2013.

[Sch14] ———, *On the Geometry of Circle Bundles with Special Holonomy*, preprint, arXiv:math.DG/1409.2741, 2014.

[Sha70] R. Shaw, *The subgroup structure of the homogeneous Lorentz group*, Quarterly Journal of Mathematics **1** (1970), no. 21, 101–124.

[Sha97] R. W. Sharpe, *Differential Geometry: Cartan's Generalization of Klein's Erlangen Program*, Graduate Texts in Mathematics, Springer, 1997.

[Tes12] G. Teschl, *Ordinary differential equations and dynamical systems*, Graduate Studies in Mathematics, vol. 140, American Mathematical Society, Providence, RI, 2012. MR 2961944

[Til70] H. Tilgner, *A class of solvable Lie groups and their relation to the canonical formalism*, Annales de l'institut Henri Poincaré (A) Physique théorique **13** (1970), no. 2, 103–127.

[Ton97] P. Tondeur, *Geometry of Foliations*, Monographs in mathematics, Springer, 1997.

[Wal50] A. G. Walker, *Canonical form for a Riemannian space with a parallel field of null planes*, The Quarterly Journal of Mathematics **1** (1950), no. 1, 69–79.

[Wal68] F. Waldhausen, *On irreducible 3-manifolds which are sufficiently large*, Annals of Mathematics (1968), 56–88.

[WR99] C. T. C. Wall and A. Ranicki, *Surgery on Compact Manifolds*, 2nd ed., Mathematical surveys and monographs, American Mathematical Society, 1999.

[Wu64] H. Wu, *On the de Rham decomposition theorem*, Illinois Journal of Mathematics **8** (1964), no. 2, 291–311.

[Yur92] U. Yurtsever, *A simple proof of geodesical completeness for compact space-times of zero curvature*, J. Math. Phys. **33** (1992), no. 4, 1295–1300. MR 1154009 (92k:53125)